DOCUMENT RESUME

ED 184 832 SE 030 443

AUTHOR	Maletsky, Evan M.: And Others
TITLE	A Brief Course in Mathematics for Junior High School Teachers, Part 1. Revised Edition.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	65
NOTE	190p.: For related document, see SE 030 444. Contains numerous light type. Best copy available.
EDRS PRICE	MF01/PC08 Plus Postage.
DESCRIPTORS	Junior High Schools: *Mathematics Education: *Mathematics Teachers: Number Concepts; Number Systems: Secondary Education: Secondary School Mathematics: *Secondary School Teachers: Set Theory: *Teacher Education: *Textbooks
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This is part one of a two-part School Mathematics Study Group (SMSG) textbook. The text serves as a means to prepare teachers to teach the SMSG text "Mathematics for Junior High School." The six chapters found in part one are: (1) Sets, (2) Numeration, (3) Computation in Bases Other than Ten, (4) Mathematical Systems, (5) Introducing New Numbers, and (6) Binary Operations. (MK)

****
* Reproductions supplied by EDRS are the best that can be made *
* from the original document. *
****

NATIONAL SCIENCE FOUNDATION
COURSE CONTENT IMPROVEMENT
PROGRAM
OFFICIAL RECORD COPY
Do Not Remove From Office

ED184832

U.S. DEPARTMENT OF HEALTH,
EDUCATION & WELFARE
NATIONAL INSTITUTE OF
EDUCATION

THIS DOCUMENT HAS BEEN REPRODUCED EXACTLY AS RECEIVED FROM THE PERSON OR ORGANIZATION ORIGINATING IT. POINTS OF VIEW OR OPINIONS STATED DO NOT NECESSARILY REPRESENT OFFICIAL NATIONAL INSTITUTE OF EDUCATION POSITION OR POLICY

"PERMISSION TO REPRODUCE THIS
MATERIAL HAS BEEN GRANTED BY
Mary L. Charles
NSF

TO THE EDUCATIONAL RESOURCES
INFORMATION CENTER (ERIC)."

SE 030 443

A BRIEF COURSE IN MATHEMATICS FOR JUNIOR HIGH SCHOOL TEACHERS

Part 1

(Revised Edition)

The following is a list of all those who participated in the preparation of this volume:

Evan M. Maletsky, Montclair State College, Upper Montclair, N. J.
Nick L. Massey, Seattle Public Schools, Seattle, Washington
Irene St. Clair, Texas Education Agency, Austin, Texas
Robert W. Scrivens, Waterford Township Schools, Pontiac, Mich.
Max A. Sobel, Montclair State College, Upper Montclair, N. J.
Marvin L. Tomber, Michigan State University, East Lansing, Mich.

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

© 1965 by The Board of Trustees of the Leland Stanford Junior University
All rights reserved
Printed in the United States of America

PREFACE

This text has been written under the auspices of the School Mathematics Study Group as a means of preparing one to teach the SMSG text Mathematics for Junior High School, Volume I. It attempts to develop the basic content necessary to understand and teach the material covered in Volume I of the junior high school series.

Also included throughout this text are comments on suggested methods of presenting this material to seventh graders. Additional helpful hints can be found in the SMSG Teacher's Commentary that accompanies Mathematics for Junior High School, Volume I. Thus, it would be quite beneficial for one to study this text concurrently with the available SMSG seventh grade materials.

Although designed specifically to accompany the aforementioned SMSG text, the material presented herein should adequately prepare one to teach any of the so-called "modern" approaches to seventh grade mathematics. Almost all of these programs have certain key features in common, such as:

(a) emphasis on the rationale of the fundamental operations;

(b) discussion of properties and structure of the number system;

(c) attention to concepts of non-metric as well as metric geometry;

(d) exploration of other systems of numeration as a device for strengthening the understanding of our own decimal system.

It has been the experience of teachers who have participated in such programs as the SMSG one that seventh grade youngsters (as well as teachers) show far more interest and enthusiasm in their studies of mathematics than is the case with traditional programs that present a heavy emphasis on computational techniques. This is not to imply that computation is neglected in the newer approaches; rather it is developed with careful attention paid to meaning and understanding.

In this text, class exercises are interspersed throughout, with answers given at the conclusion of each chapter. Answers to the end of chapter exercises are to be found at the end of the book. The exercises should be completed as soon as the material is read in order to strengthen ideas presented within each section. Furthermore, each chapter closes with an additional collection of exercises to provide practice of key ideas. A series of masters are available for preparing projectuals to use in conjunction with the teaching of a course based on this book.

This text was written with the thought that it would be used in an in-service course for which there would be an instructor or consultant available. However, sufficient details have been presented throughout so that a teacher should be able to master the material independently.

Although these units are based on Volume I of *Mathematics for Junior High School*, it was necessary to present some ideas that first appear in Volume II in order to provide a complete picture in some areas. Thus, the set of real numbers is discussed here although they are not formally treated until the eighth grade in most texts.

TABLE OF CONTENTS

Chapter Page

INTRODUCTION v

1. SETS 1
1.1 The Concept of Sets 1
1.2 Relations Between Sets. 4
1.3 Intersection and Union of Sets. 7
1.4 Sentences, the Number Line, and Truth Sets. 10
1.5 Compound Number Sentences 14
1.6 Conclusion. 17

2. NUMERATION 25
2.1 Early Numeration Systems. 25
2.2 Expanded Notation and Exponents 29
2.3 Numeration in Other Bases 34
2.4 Changing from One Number Base to Another. 40
2.5 Just For Fun. 44

3. COMPUTATION IN BASES OTHER THAN TEN. 51
3.1 Addition. 51
3.2 Subtraction 57
3.3 Multiplication. 61
3.4 Division. 66

4. MATHEMATICAL SYSTEMS 75
4.1 Binary Operation. 75
4.2 A Mathematical System 77
4.3 Mathematical Systems - Additional Properties. 83
4.4 Clock Arithmetic. 87
4.5 Conclusion. 92

5. INTRODUCING NEW NUMBERS. 97
5.1 The Counting Numbers and the Whole Numbers. 97
5.2 Positive Rational Numbers 100
5.3 Equivalent Fractions. 103
5.4 Order 107
5.5 Whole Numbers and Rational Numbers. 110
5.6 The Integers. 112
5.7 Ordered Pairs 117
5.8 Historical Note 120

Chapter Page

6. BINARY OPERATIONS. 127
6.1 Addition. 128
6.2 Properties of Addition. 133
6.3 Multiplication. 137
6.4 Properties of Multiplication. 139
6.5 The Distributive Property 141
6.6 Subtraction 142
6.7 Division. 145
6.8 Operations on the Integers. 148

ANSWERS TO CHAPTER EXERCISES

INTRODUCTION

Mathematics is concerned with many things; some serious and some frivolous, some hard and some easy. The computation of batting averages is mathematics. The study of surfaces which may be pulled into the shape of spheres is mathematics. The solution of the well-known problem,

$$\begin{array}{r} \text{SEND} \\ +\ \text{MORE} \\ \hline \text{MONEY} \end{array}$$

where each letter is to be replaced uniquely by a digit to form a correct addition problem, is mathematics. Some very simple but careful reasoning will produce the answer. In this text we shall study some of the branches of mathematics and lay a foundation for further study.

The diversity in mathematics may be compared to the diversity available in reading. We may choose our reading in many ways. We may read for recreation or for knowledge. This analogy may be carried on in other ways. Just as some read on rare occasions, others may read compulsively, being uncomfortable when they are more than three feet from a book, paper, or magazine. Whatever the reason or whatever the level, nearly everyone finds something of interest to read. Certainly, everyone who can read finds the ability to read valuable. The same may be said for mathematics: There is something of interest or value to everyone. There are those who will use mathematics to verify their paychecks and there are some who are compulsive mathematicians, only happy when thinking of mathematics.

The analogy may be extended in still other directions. No one person is able to read all the books, magazines, pamphlets, and papers published, and no one individual can be knowledgeable in all areas of mathematics. There are those who read and also write; in mathematics there are those who study, those who use mathematics, and those who go further and create new mathematics.

Every discipline has a vocabulary of its own. This may include special words, such as magnetohydrodynamics, or common words with meanings specialized to the subject, such as function. This is usually an attempt to achieve precision and economy in communication. Unfortunately a jargon is sometimes introduced in a discipline to mask a fundamental lack of knowledge and to appear scientific.

To achieve economy and precision in mathematics some very common words are used to convey deep ideas. The word "number," as used in mathematics, briefly conveys a very abstract idea. To the layman, the word "number" brings to mind some symbols. We recognize that the symbol is not the concept. When we write the word "horse" we think of a "solid-hoofed animal used for riding on or drawing burdens," but the word "horse" is not the animal but is rather a symbol for the animal. When we write the symbols 6, VI, and $\frac{24}{4}$, we are writing various forms of names for a number concept. To be precise we speak of the symbols 6, VI, and $\frac{24}{4}$ as numerals which name the same number. There are, of course, situations in which this degree of precision is necessary and other occasions when this precision becomes pedantic. A physician, when speaking to a colleague, may refer to a patient's broken tibia while the patient is content to speak of a broken leg.

We might speak of the numbers whose numerals are 1, 2, 3, and 4. This would be perfectly correct. However, we frequently choose to write "the numbers 1, 2, 3, and 4" and trust the context will make our meaning clear.

It is desirable that pupils know the distinction between numeral and number. For example, we may wish to write:

$$\frac{4[(3 \times 3) + 3]}{6} = 8 .$$

On the face of it, it is ridiculous to claim the two sides of this expression are the same. When we realize these are two names which name the same number the statement becomes meaningful.

The purpose of the introduction in this text is to illustrate that there is more to mathematics than computing and that there is much of interest in mathematics that may readily be demonstrated to the junior high school student. The aims of an introductory chapter at the beginning of the seventh grade are many. It is worthwhile to reawaken an interest in mathematics, to show the power of mathematics in many varied situations, and to give an indication of some of the things to come.

Chapter 1 in Volume I of Mathematics for Junior High School illustrates how this might be done. In teaching such a chapter it may not be best to do the entire chapter at one time. A part of such a chapter could be covered until the student's interest is captured and the rest postponed until later. A good time to return to such material is just before vacations.

A Number Game

The interest of the student might be aroused by playing a very simple number game. Here is a game with simple rules for two players. To describe the game we will call the players A and B. From among the numbers 1, 2, 3, 4, 5, 6 player A picks a number. Player B then picks a number, again from 1, 2, 3, 4, 5, 6 and adds it to the number A picked. It is now A's turn. He picks a number from the six and adds it to the preceding sum. The game continues in this fashion. The game is won by the player who is able on his turn to pick the number, from 1, 2, 3, 4, 5, 6, which makes the total sum 56. The same number may be picked as many times as desired.

A Sample Game

A chooses 6	B chooses 5; 6 + 5 = 11
A chooses 3; 11 + 3 = 14	B chooses 6; 14 + 6 = 20
A chooses 1; 20 + 1 = 21	B chooses 4; 21 + 4 = 25
A chooses 6; 25 + 6 = 31	B chooses 3; 31 + 3 = 34
A chooses 2; 34 + 2 = 36	B chooses 6; 36 + 6 = 42
A chooses 3; 42 + 3 = 45	B chooses 4; 45 + 4 = 49
A chooses 4; 49 + 4 = 53	B chooses 3; 53 + 3 = 56

B wins!

Pair off the class members and play this game. Can you find a pattern that will enable you to always win this game?

Is this mathematics? We would say it is, for reason and deduction allows you to answer the above questions. This game has a feature that is desirable in the classroom; it may be varied. For example, the game may be played with the numbers 1, 2, 3, 4, 5, 6, 7 and winning sum 85. There are, of course, many variations.

This game is an example of a mathematical problem (or puzzle) which may be solved without any formal knowledge of mathematics. It is amusing to play and it is pleasurable to discover the strategy of the game. Since there are variations in which the winning strategy is not much changed, many students have an opportunity to make a discovery.

In the remainder of this introduction we shall examine in detail some problems which are typical of those given in Chapter 1 of Mathematics for Junior High School, Volume I. There are many such problems; their object is to increase interest in mathematics. They should not be allowed to become frustrating. A simple question at the right time may lead the student or class in the right direction.

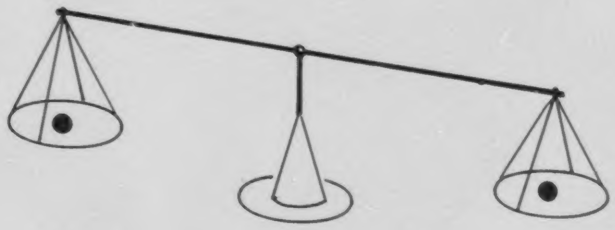
Weighing Problems

Consider the following reasoning problem:

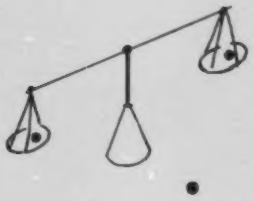
> Eight marbles all have the same size, color, and shape. Seven of them have the same weight and the other is heavier. Using a balance scale, how would you find the heavy marble if you make only two weighings?

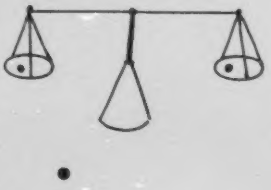
There are many ways one might begin to work on this problem. Many students, however, hesitate to begin because of the relatively large number of marbles involved. This difficulty occurs in many problems and students should be encouraged to invent similar but easier problems to be solved first. The eight-marble problem is a good example of a problem in which this may easily be done.

A student might begin with this problem: Two marbles are identical in appearance. One marble is heavier than the other. Determine which marble is the heavy one using a beam balance. The solution of this problem is of course very easy.



Now one might be encouraged to look at the problem of three marbles. Is it possible to determine the heavy marble with one weighing? The student should now come to the conclusion that an equal number of marbles must be placed in each pan (tray) of the beam balance. He also will see that there is a need to consider cases.





While it is not recommended that the students work their way case by case up to the case of eight marbles, a few more cases may be helpful. Can the heavy marble among four marbles always be determined in one weighing? In two weighings? Having done the problem for four marbles, the problem of five marbles is easily done.

Studying the cases with a smaller number of marbles has served the purpose of making the problem seem less formidable and gives suggestions for doing the problem of eight marbles. We must be careful, however, that our special cases do not mislead us. It may appear for the cases already done that there is a different way to start in the case of an even number of marbles and an odd number of marbles. To do the problem of eight marbles, it may seem that the first weighing should be four marbles against four marbles. This would surely tell us which collection of four marbles contains the heavy marble. We know from the exploratory problems that it takes two weighings to determine a heavy marble from among four marbles. This approach to the problem would thus require three weighings. If it is possible to do the problem as stated, this is not the way to do it!

The elementary distinction between even and odd which may seem to indicate a general pattern does not in this case reveal the true pattern. It may happen as it did here, that a small number of cases seem to indicate a pattern which turns out not to be the correct pattern. If you have safely passed cars on a certain hill three times, can you make any valid assertions about what will happen the fourth time? Of course, the true pattern may be discovered with a small number of experiments.

To continue the solution, we do know that if the heavy marble is known to be among three marbles or among two marbles it may be located with one weighing. These remarks should allow the reader to complete the problem.

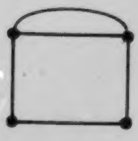
It is by no means implied that this problem must be worked by this sequence of discovery steps. What has been shown is that a seemingly complex problem may sometimes be done by exploration through simple problems. It is important for students ultimately to produce correct answers and equally important that they not be forced into the same mode of thought as their teachers.

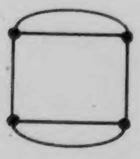
This class of problems was proposed originally as counterfeit coin problems; an example is given below:

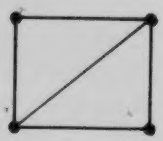
Among six coins identical in appearance there is one counterfeit coin. It is known that the counterfeit is made from impure metal and does not weigh the same as the genuine coins. What is the smallest number of weighings with a beam balance which would be required to locate the counterfeit coin? Will the answer change with additional requirements? Are additional weighings required to determine whether the false coin is too heavy or too light?

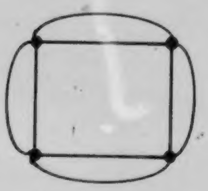
Unicursal Problems

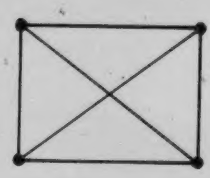
Most children have worked at problems which mathematicians call Unicursal problems. A figure is given composed of segments, either line segments or curves, and the player is required to trace the figure without lifting his pencil and without retracing a segment which has already been covered. Try this with the following figures.





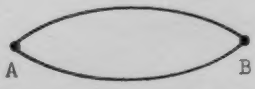




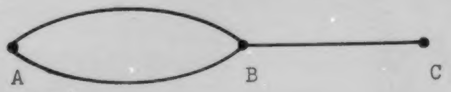


Anyone who has attempted one of the puzzles will be surprised to learn that the key to understanding them is mathematics. Again, no involved mathematics is required. Unicursal problems are another good illustration of the power and versatility of mathematics.

Again, let us begin by examining some simple problems. Let us see how a student might be encouraged to make discoveries.

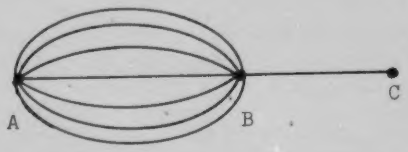


The figure above is easily traced according to the rules. Also the figure

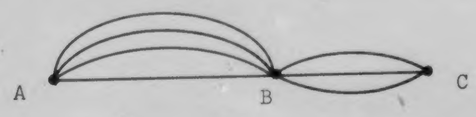


may be traced without violating the rules. In this case a little care must be exercised. A tracing which obeys the rules of the game, or what we will call a successful tracing, cannot begin at A but must begin at B or C. The student should put into words a reason for this statement. That is, why will it be impossible to complete a successful tracing if begun at A?

The figure below may also be traced according to the rules.



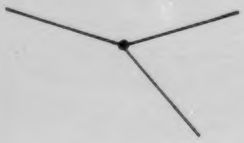
Why can't a successful tracing begin at B ? Why must the tracing begin or end at A or begin or end at C ?



Is it possible to trace the figure above? If it is possible, may you start at any point?

We would like to make some general statements about these problems, if possible. The rule which says the pencil is not to be lifted from the paper tells us that a figure composed of two disjoint parts cannot be traced successfully. To use a technical word: Any figure which can be traced according to the rules must be connected. In the last four figures, do the answers concerning tracing depend on the number of segments meeting at a point?

The examples indicate that the solution may not depend upon the total number of segments at a point; rather, the examples indicate a difference according to the parity of the number of segments at a point. Parity refers to the property of being odd or even. The examples suggest that a successful tracing of a figure with an odd number of segments at a point will begin or end at that point. For a point with one segment (part of a larger figure) it is clear that the tracing must start or end at that point. Let us think about a point with three segments.



(part of a larger figure)

Suppose we do not begin at this point, then in the course of a tracing which is to be successful we must come into this point, thereby covering one of the segments. There remain two segments not yet covered. A successful tracing must continue by leaving the point over one of the segments. There remains one segment to be traced. Once the third segment is traced we are finished, for there is no way then to move away from the point. That is, a point with three segments which is not a starting point of a successful tracing must be the finishing point.

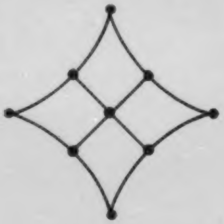
This reasoning may easily be extended to cover the case of any point with an odd number of segments. A point with an odd number of segments which is not a starting point of a successful tracing must be the end point; the tracing must stop at the point.

Let us review this conclusion. We look at an odd point, a point with an odd number of segments, and conclude that if it is not a starting point of a successful tracing then it is the end point. This does not eliminate the possibility that an odd point may be a starting point. We may say that an odd point which is not an end point is a starting point. We may classify the points of a successful tracing as: starting point, intermediate points, and end point. An odd point which is not the starting point must be the end point. An odd point can not be an intermediate point. Thus an odd point which is not the end point must be the starting point.

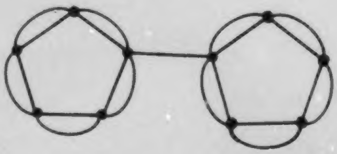
In a possible tracing problem each odd point must be a starting point or end point. Since, according to the rules, there can be at most one starting point and one end point, a figure which has more than two odd points cannot be traced successfully.

Is it possible to trace these figures without lifting your pencil and without retracing any segments?

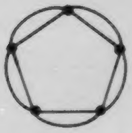
(a).



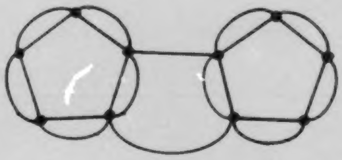
(c).



(b).

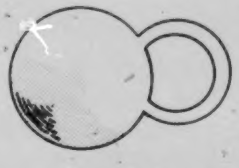


(d).



Unicursal problems, or tracing problems, have an honorable place in mathematics. The mathematician, Euler, (1707-1783) was the first to systematically study these puzzles in connection with the Koenigsberg bridge problem. Euler was a prolific mathematician; his collected works are still being edited. It is estimated that over sixty large volumes will be required. Among his many interests were the properties of figures in space.

sphere



sphere with a handle

For example, Euler was the first to give a mathematical way of differentiating between these two figures without saying "sphere with a handle". Euler's solving of this problem and the Unicursal problem was a part of the beginning of that branch of mathematics known as topology.

Euler wrote many mathematical texts. It has been claimed that until the recent flurry of the new texts, every high school mathematics text was a revision of Euler's texts.

Counting Problems

In a school class of 35 pupils, all the pupils take either French or German; 21 students are enrolled in French and 17 are enrolled in German. How many students are enrolled in both French and German?

This is an example of one form of counting problem. Let us analyze this problem. There must be some students enrolled in both French and German, for otherwise there would have to be $38 = 21 + 17$ students in the class. When we add 21 and 17 we are counting among the 21 those students who are taking only French and those who are taking both French and German. Among the 17 we again count those students who are taking both French and German. The sum, $38 = 17 + 21$, represents the number of those students taking only French, those students taking only German, and twice the number of students taking both French and German. The number 35 is the sum of the number of those taking only French, those taking only German; and the number taking both French and German.

$$\text{French} + \text{German} + \text{Both} = 35$$
$$\text{French} + \text{German} + 2 \times \text{Both} = 38$$

Now we see that the number taking both languages must be 3. From this we may compute, if we wish, the number taking only French, the number taking only German.

As a somewhat more complicated example of the same sort of problem we have:

> In a class of 225 students, all of whom are required to take a foreign language, 94 students are enrolled in French, 128 students are enrolled in German, and 88 are taking Russian. No other foreign languages are offered. It is also known that 36 are enrolled in French and German, 22 in German and Russian, and 29 in French and Russian. Are any students taking three languages? If so, how many?

The similarity between this problem and the preceding ones is clear. The similarity may suggest that we begin the problem as before. If we add 94, 128, and 88 to obtain 310, we find we have accounted for the students who are taking two languages twice. That is, a student who is taking French and Russian has been counted in the 94 students taking French and again in the 88 students taking Russian. Thus, it is not surprising that the sum, 310, exceeds the total number of students, 225. Suppose we attempt to represent the total number of students in terms of the number taking the various languages.

Thus, from $310 = 94 + 128 + 88$ we must subtract the duplications. With some care the student may complete the solution in this fashion.

Caution: Subtraction to eliminate the duplication may result in other duplications. (There are exactly 2 students taking all three languages.)

In the next chapters we will see how, with the aid of a little mathematical notation and knowledge, the reasoning needed to do this problem may be much simplified.

Some of the many facets of mathematics have been introduced in the problems we have discussed. As examples of some other aspects of mathematics to come we will list some problems which you may think about and even solve now but whose solutions will be natural consequences of the material to be studied later. Answers to these and the other problems presented in this introduction are included at the end of the text with the chapter exercise answers.

> There are three houses on a street. At the curb there are three utilities; water, electricity, and gas. Is it possible to connect each utility to each house without the connections crossing each other?
>
> Objects are to be weighed on a balance scale by comparing them with standard weights. If you wish to weigh objects, in pounds, between 1 pound and 63 pounds, what would be the most efficient set of standard weights? (Efficient means the smallest possible number of weights.)

Chapter 1

SETS

Introduction

Several questions usually arise among mathematicians, educators, pupils and parents about the pedagogical soundness of the teaching of sets and set language. Questions are raised as to why, where, when, and how sets should be introduced in the seventh grade curriculum. Some argue that a separate chapter should be included; some say that concepts of sets should be introduced as they are needed; and some educators claim that sets are not needed at all to be "modern."

There is merit in each of these viewpoints, but in this book we will take the position that for the junior high school youngsters set language should be presented primarily as it is needed to clarify mathematical concepts. The reason that we are including these concepts in a separate chapter in this text is that, because of the limited time a teacher has available to spend on an in-service program of this nature, familiarity with set language will expedite our presentation of other mathematical ideas appearing in later chapters. The language of sets will give us a precise way of talking about certain number ideas, properties of operations, and geometrical concepts.

1.1 The Concept of Sets

We say that a set is a well-defined collection of objects. What is meant by this? Certainly we know what is meant by a collection: A bunch of bananas, a herd of elephants, a set of dishes, the things on my desk, and so on. When we say it is well-defined, we must be certain that the description allows us to determine without ambiguity whether or not an element belongs to the set. The objects in a set need not be related in any way except that we treat them as a single group. For example, the set consisting of the number 5, the word "Tuesday," and the moon is a well-defined collection. However, in mathematics we usually speak of sets with elements that have some property in common. For example, the set of whole numbers, the set of primes, or the set of points on a line.

20

There are many ways of describing a set. For example, each of the following describe the same set:

The set of whole numbers between 2 and 12.
The set of whole numbers from 7 through 11, inclusive.
The numbers 7, 8, 9, 10, and 11.
$\{7, 8, 9, 10, 11\}$
$\{8, 10, 11, 9, 7\}$

Notice the use of braces, $\{\ \}$, with the members or elements of the set included between them. Frequently, an arbitrary capital letter is used to name the set.

$$M = \{7, 8, 9, 10, 11\}$$

The "things" in a set need not be objects you can touch or see. The set of all Beethoven symphonies does not contain any concrete objects. You may have heard some of its members, however. The set of all football teams in the United States is a set whose members are themselves sets of players.

Sometimes the symbol "ϵ" (stylized Greek letter, epsilon) is used to mean "is a member of," or "is an element of." Thus we can express the fact that the number 8 is a member of set M above by writing:

$$8 \in M .$$

We can express the fact that the number 6 is not a member of set M by writing

$$6 \notin M .$$

At times we encounter a set which contains no members. Such a set is called the "null set" or the "empty set," and is designated by $\{\ \}$ or $\emptyset$. If set B is the set of all odd whole numbers less than 1, then set B has no members and we can write $B = \emptyset$. Another example of the empty set is the set of United States cities located in the province of Manitoba, Canada.

Often it is inconvenient to list all the members of a set within braces. The set of letters of the English alphabet could be shown as $E = \{a,b,c,\ldots,z\}$. Here a pattern has been established and the three dots mean "and so on in like manner" to z. The set of whole numbers may be shown as $W = \{0,1,2,3,4,\ldots\}$. The fact that no element is named after the ellipsis (...) implies that the listing of elements does not terminate but rather continues on in the same pattern without end. Such a set is called an infinite set. A finite set is a set which may be counted with the counting coming to an end. Set E above is an example of a finite set while set W illustrates an infinite set.

Some other examples of finite sets are:

$P = \{2,4,6,8,10\}$;
$Q = \{3,6,9,\ldots,81\}$;
the empty set, $\emptyset$;
the set of people in the United States of America.

Some additional infinite sets are:

$T = \{5,10,15,\ldots\}$;
the multiples of 257;
the points on a line;
the set of prime numbers.

Class Exercises

1. Tell whether or not each of the following sets is well-defined.
   a. The set of states of the United States bordering the Pacific Ocean.
   b. The set of small states in the United States.
   c. The set of all whole numbers which are not multiples of 3.
   d. The set of all whole numbers between 0 and 1.
   e. The letters which are in the name of your school and not in your last name.
2. Describe each of the following sets in at least two other ways:
   a. All odd whole numbers from 1 to 12 inclusive.
   b. $M = \{10,20,30,\ldots,100\}$.
   c. The set of integers greater than 50.
   d. The set of whole numbers between 20 and 30 and greater than 50.
3. Tell whether or not each of the following is true or false and explain your reasoning.
   a. $3 \in \{2,3,4,5\}$
   b. $\{0\} = \emptyset$
   c. $\{\emptyset\} = \emptyset$
   d. $17 \notin \{5,6,7,8,\ldots\}$
   e. $\{e,f,d\} \neq \{f,e,d\}$
   f. $32 \notin \{4,8,12,\ldots,96\}$

4. Classify the following sets as finite or infinite.

   a. Set of all whole numbers which are multiples of 3.
   b. Set of all numbers x such that $x + 1 = x$.
   c. Set of grains of sand on the beach of Coney Island.
   d. Set of all positive integers smaller than 0.
   e. All mathematics textbooks in the United States.

5. Let $M = \{3,5,7,\ldots,23\}$. What are the elements of this set? (Beware.)

1.2 *Relations Between Sets*

Consider the set of the first three letters of the alphabet, $A = \{a,b,c\}$, and the set containing the letters of the word *cab*, $B = \{c,a,b\}$. Since the order in which the members of a set are listed is immaterial, we can say that these two sets are *identical* or *equal*. This can be written as $A = B$. (Remember "=" here means "names precisely the same thing.")

Think of the sets $A = \{a,b,c,\ldots,z\}$ and $C = \{1,2,3,\ldots,26\}$. A matching or *one-to-one correspondence* may be illustrated between these two sets as follows:

$$\begin{array}{l} A = \{a,b,c,\ldots,z\} \\ \quad\;\; \updownarrow\,\updownarrow\,\updownarrow \quad\;\; \updownarrow \\ C = \{1,2,3,\ldots,26\}\,. \end{array}$$

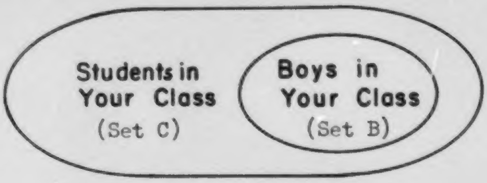
A one-to-one correspondence associates each element of set A to an element of set C and each element of set C to an element of set A. Obviously, other matchings are possible with the same two sets.

Certainly set A is not equal to set C, $A \neq C$, since they do not have the same elements. However, they do have the same *cardinality*; that is, the same *number* of members. Therefore, we say that set A is *equivalent* to set C. The equivalence of two sets is frequently written as: $A \longleftrightarrow C$. Remember, two sets are equivalent if the elements of each can be put in a one-to-one correspondence.

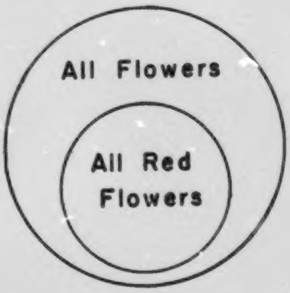
It should be apparent from the definitions that all equal sets are equivalent, but not all equivalent sets are equal.

If two sets have no members in common, we say they are *disjoint*. For example, consider the sets $R = \{6,8,12,14\}$ and $S = \{5,7,9\}$. Note that R and S have no common members. Therefore, we say that R and S are disjoint sets.

Think of the set of members of your class, C. The set of boys in your class, B, is a subset of the set of members of your class. This may be represented by drawing a sketch, often called a "Venn" diagram.

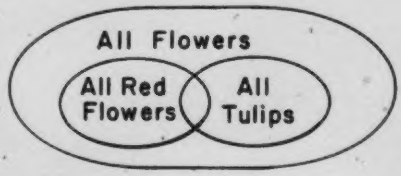


To write this relationship in mathematical language we use the symbol "$\subset$" which may be read "is a subset of" or "is contained in." You can now write: $B \subset C$.



The diagram at the right illustrates that the set of all red flowers is a subset of the set of all flowers. Let the set of all red flowers be called R and the set of all flowers be called F. The relationship of R and F can then be written as: $R \subset F$.

Note in the following Venn diagram that the set of all red flowers belongs to the set of all flowers, and that the set of all tulips also belongs to the set of all flowers:



Let the set of all tulips be called T. The above relationship may now be expressed as:

$R \subset F$, and
$T \subset F$.

What can you say about the relationship of sets R and T? You would certainly have to say that some tulips are red and are thus contained in set R. This is why the sets R and T are shown as overlapping ovals in the diagram. But you certainly cannot say that T   R is true. Why not?

As another example let us find all the subsets of $B = \{1,3,5\}$. They would be: $\{1\}$, $\{3\}$, $\{5\}$, $\{1,3\}$, $\{1,5\}$, $\{3,5\}$, $\{1,3,5\}$, and the empty set, $\emptyset$. Any set is a subset of itself, and the empty set is considered to be a subset of every set. This may be a little clearer if you consider the set {Tom, Dick, Harry}, where we now think of the set of three boys whose names are Tom, Dick, and Harry, and not the set of three words--"Tom," "Dick," and "Harry." We now ask: "In how many ways could you ask none or some of the three boys to go to the ball game with you?" The answer is that you could ask any one of them, or any two of them, or all three of them, or none of them. Thus, the subsets are: {Tom}, {Dick}, {Harry}, {Tom, Dick}, {Tom, Harry}, {Dick, Harry}, {Tom, Dick, Harry}, and $\emptyset$.

We can state this concept of a subset in mathematical language as follows:

> If every element of a set S belongs to a set T, then S is said to be a *subset* of T. We say that S is contained in T; that is, $S \subset T$.

Also,

> S is a proper subset of T if $S \subset T$, $S \neq T$.

For example, the proper subsets of set $B = \{1,3,5\}$ would be all of the subsets of B except B itself; namely: $\emptyset$, $\{1\}$, $\{3\}$, $\{5\}$, $\{1,3\}$, $\{1,5\}$, and $\{3,5\}$.

Sometimes the symbol $\subseteq$ is used to represent "is a subset of" and the symbol $\subset$ used only to represent "is a proper subset of."

Class Exercises

6. Draw Venn diagrams illustrating the following relationships:
   a. B is a proper subset of A.
   b. B and D are proper subsets of A, and B and D are disjoint.
   c. B and C are proper subsets of A, and B and C are *not* disjoint.

7. Given the sets $S = \{0,5,7,9\}$ and $T = \{0,2,4,6,8,10\}$.
   a. Find K, the set of all numbers belonging to both S and T. Is K a subset of S ? of T ? Draw a Venn diagram illustrating this.

b. Find M, the set of all numbers each of which belongs to S or to T or to both. (We never include the same number more than once in a set.) Is M a subset of S ? Is T a subset of M ? Is M finite?

c. Find R, the subset of M, which contains all the odd numbers in M. Of which others of our sets is this a subset?

8. The following table has been started:

	Set	Number of members	Subsets	Number of Subsets
a.	∅	0	∅	1
b.	{△}	1	∅, {△}	2
c.	{△, ○}	2	∅, {△}, {○}, {△,○}	4
d.	{△, ○, □}	3		
e.	{△,○,□,☆}			

How many different subsets can be formed from the members of the set in d ? From the members of the set in e ? Try to predict how many different subsets a set with eight members would have.

1.3 Intersection and Union of Sets

We often think of elements common to two sets. Suppose that in your class you asked all the boys who play in the school band to stand. Let this be the following set:

B = {Bill, Jim, Tom, Sam}.

Suppose these boys then sat down and you asked all the boys with red hair to stand. Let this be the following set:

R = {Sam, Tom, Carl}.

Finally, suppose you asked all the red-haired band members to stand. What would this set be? It would be the set:

{Tom, Sam}.

This set is called the intersection of set B and set R. The combining of two sets in this manner is an operation on these sets.

> The intersection (symbol, $\cap$) of two sets is the set of all elements common to each of the given sets.

Let us consider two other sets, G and H, defined as follows:

$$G = \{4,5,6,7,8,9\};$$
$$H = \{2,4,6,8,10\}.$$

From these two sets another set, K, may be formed whose members appear in both G and H:

$$K = \{4,6,8\}.$$

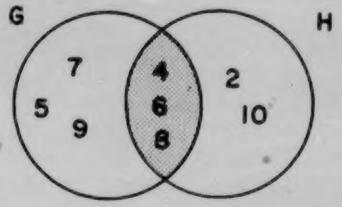
Set K consists of the members that sets G and H have in common and is therefore the intersection of the two sets. This may be written as:

$$\{4,5,6,7,8,9\} \cap \{2,4,6,8,10\} = \{4,6,8\}$$

or

$$G \cap H = K.$$

A diagram may also be used to illustrate this idea:



$G \cap H = \{4, 6, 8\}$

The shaded region indicates the intersection of the two sets.

Now consider set $R = \{1,2,3,4,5\}$ and set $S = \{6,7,8,9\}$. Sets R and S have no members in common (i.e., they are disjoint sets). Therefore, the intersection of the two sets is the empty set and we write $R \cap S = \emptyset$. Draw a Venn diagram illustrating this case.

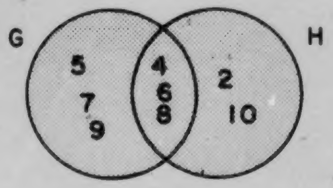
Another operation on sets is the combining of two sets in such a way that each of the members of the new set is in at least one of the two given sets. Recall again the members of the band and the red-haired boys. If all the boys who were either in the band or who had red hair were asked to stand, we would have the set:

{Bill, Carl, Jim, Tom, Sam}.

This is called the union of these two sets.

> The union (symbol, $\cup$) of two sets is the set of all elements that are in at least one of the given sets.

As another example, consider again set $G = \{4,5,6,7,8,9\}$ and set $H = \{2,4,6,8,10\}$. The union of set G and set H (written $G \cup H$) would be $\{2,4,5,6,7,8,9,10\}$ which we shall designate as set L. Therefore, $G \cup H = L$. A diagram such as the following may be drawn to illustrate this idea:



$G \cup H = L$

The shaded region shows the union of the two sets. (Remember, there is only one number 4, one 6 and one 8, therefore, 4, 6, and 8 are included only once in the union.)

Recall again set $R = \{1,2,3,4,5\}$ and set $S = \{6,7,8,9\}$. Then

$$R \cup S = \{1,2,3,4,5,6,7,8,9\}.$$

Can you illustrate this with a Venn diagram?

We would like to interject a note here again that much of this chapter is being presented as background information for teachers, and that most textbooks for students probably integrate these concepts as they are needed to develop some mathematical idea. However, some parts of this material may be presented to a class as a little side trip. Most youngsters enjoy this as something different and fairly easy to grasp.

As these ideas are presented many visual aids may be used. Sets of objects, plastic containers, and the use of overhead projectuals adapt themselves readily to this area. Different colored sheets of acetate cut in various shaped and placed on the stage of the overhead projector depict clearly the intersection and union operations. The student needs to be led to discover some mathematics for himself, and this topic is one in which this may be done quite effectively.

Class Exercises

9. Given: $A = \{b,d,e,f\}$
$B = \{a,b,c,d,e,f\}$
$C = \{a,b,c,e\}$
$D = \{a,c\}$
$E = \{d,f\}$
$F = \{b,e\}$

Which of the following sentences are true?

(a) $A \cup C = B$
(b) $B \cup C = B$
(c) $A \cup B = C$
(d) $B \subset C$
(e) $B \cap C = C$
(f) $F \subset F$
(g) $E \cup F = A$
(h) $A \cap C = F$
(i) $D \subset C$
(j) $(E \cup F) \subset (E \cap F)$
(k) $(D \cap E) \subset (D \cup E)$
(l) $D \cap E = F$

10. Let W = the set of all whole numbers.
E = the set of all even numbers.
O = the set of all odd numbers.

Describe each of the following sets:

(a) $E \cap O$
(b) $O \cup E$
(c) $W \cup E$
(d) $(E \cap O) \cup W$
(e) $(O \cap W) \cap E$
(f) $W \cup (E \cup O)$

1.4 Sentences, the Number Line, and Truth Sets

We need to pause for a moment in our discussion of sets and set language to talk about language in general, then see one of the applications for the use of sets. The teaching of mathematics not only must give the student a glimpse of the structure of the subject but must also treat the language with great care. The difference between words like "and" and "or," "if" and "only if," and "not" and "none," can mean the difference between understanding and misunderstanding.

Language also involves choice of descriptive words. Unlike the chemist, who uses long compound words to describe his materials, the mathematician often selects common words to describe uncommon concepts. The teacher should beware of dictionary meanings for words such as rational, real, imaginary, group, field, limit, term, factor, range. When these words are used as mathematical terms, they do *not* have the meanings commonly ascribed to them.

On the other hand, teachers of junior high school youngsters need to be careful of what is expected from their students in the way of verbal responses. Certainly textbooks and teachers need to be precise in their language, but perhaps the mind of a seventh grader has not sufficiently matured to enable him to make statements in as precise mathematical language as we would wish. This is one of the things we are trying to train him to do! We must keep in mind the following question: "Are we communicating with our students, and are they communicating with each other?"

Mathematicians now make use of the structure of English sentences to communicate mathematical concepts. For example, the English sentence, "He was the first president of the United States" is neither true nor false until we give a replacement for "He." This sentence is called an *open* sentence. It may be *true*: "George Washington was the first president of the United States," or *false*: "Abraham Lincoln was the first president of the United States." In fact, "$\square$ was the first president of the United States" may be a test question requiring the name of the man for which it would be a true sentence. Open number sentences are the basis of a great deal of work in mathematics. For example, consider the following mathematical sentences:

(a) $3 + 5 = 8$
(b) $9 - 2 = 6$
(c) $5 + \square = 7$
(d) $2n + 6 = 9$

Sentence (a) is a true mathematical sentence, (b) is a false sentence, and (c) and (d) are open mathematical sentences, being neither true nor false.

All sentences require verbs. Some of the most common ones in mathematics are listed in the table below.

Symbol	Verb	Example
$=$	"is equal to"	$3 + 4 = 7$
$\neq$	"is not equal to"	$5 - 2 \neq 4$
$>$	"is more than"	$7 - 3 > 1$
$<$	"is less than"	$5 < 10$
$\geq$	"is more than or equal to"	$9 \geq$ any one-digit number
$\leq$	"is less than or equal to"	$0 \leq$ any whole number

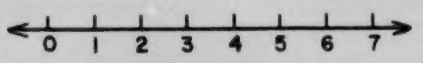
None of the examples listed above is an open sentence. They all make true statements about specific numbers which are described or represented by a single numeral such as 7 or by a mathematical or number phrase such as $3 + 4$. If we want to write an open number sentence, we need to use an open number phrase such as $\square + 7$ or $17 - \square$, where the symbol $\square$ is used to help you remember that the empty space is to be filled by some numeral from a given set. Because symbols like $\square$ are awkward to type or write, we frequently use letters such as n, a, x, or y for the same purpose. Thus, a simple open number phrase may be written as $n + 7$ instead of $\square + 7$, and an open number sentence as $n + 7 = 10$. What whole number or numbers will now make this open sentence a true statement? In this case the answer is easily obtained by trial: $3 + 7 = 10$, while $0 + 7 \neq 10$, $1 + 7 \neq 10$, $2 + 7 \neq 10$, $4 + 7 \neq 10$. We see that 3 is the only number which does the trick. It is the only replacement for n that will make the sentence true.

What whole number or numbers will make the open sentence $x < 5$ a true statement? Again, by trial we find that $0 < 5$, $1 < 5$, $2 < 5$, $3 < 5$, and $4 < 5$ are true statements while $5 < 5$, $6 < 5$, $7 < 5$, and so on, are false statements. Thus, we see that from the set of whole numbers $\{0,1,2,3,4,5,6,\ldots\}$ only the members of the set $\{0,1,2,3,4\}$ make the statement true.

What about the open sentence $n + 7 < 11$? We can translate the sentence into words by saying "the sum of a certain number and 7 is less than 11." The whole numbers which make this a true statement of inequality are the members of the set $\{0,1,2,3\}$. This set of whole numbers is called the truth set or solution set of the open sentence $n + 7 < 11$. Sentences with the verb "$=$" are called equations, whereas sentences with any of the other verbs listed above are called inequalities.

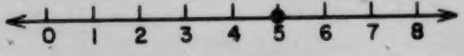
Another very useful device in our study of number sentences is to establish a one-to-one correspondence between the set of whole numbers, $W = \{0,1,2,3,4,\ldots\}$, and a set of certain points on a line. In a later chapter we will associate all the points on a line with the set R of all real numbers. We simply draw a picture of a line with arrows on both ends.

Starting at an arbitrary point that we label 0, we mark off equally-spaced intervals that are labeled with the set of whole numbers in order:



We call the number corresponding to a point the coordinate of that point. The order of the whole numbers shows up clearly by the position of the marks; $5 > 3$ indicates that the coordinate of 5 is to the right of the coordinate of 3.

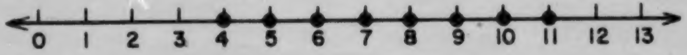
Now a picture of a solution set using the number line can be drawn. Consider the open sentence $x + 3 = 8$. This open sentence has only the one solution, 5. (A solution is an element of the solution set.) Thus, the solution set is $\{5\}$. On the number line this solution can be represented as shown below:



$x + 3 = 8$

The solution, 5, is indicated by a solid dot.

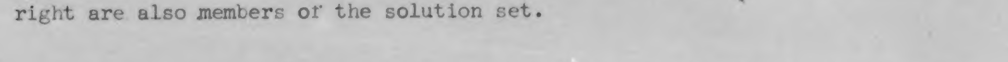
In the examples that follow we shall restrict our discussion to whole numbers only. The solution set of the inequality $n - 4 \leq 7$ can then be represented thus:



$n - 4 \leq 7$

Notice that if n represented the number 3, 2, 1, or 0, then the open number phrase $n - 4$ would represent a negative integer. Since we have restricted our discussion to the whole numbers only, these numbers are not considered as part of the solution set.

Note that on the number line we indicate the solution set by heavy solid dots. The solution set of $n - 4 > 7$ cannot be completely represented this way because it consists of all whole numbers greater than 11. However, we can represent it by heavy dots up to the arrow and the word "incomplete" to show that all the whole numbers represented by points still further to the right are also members of the solution set.



$n - 4 > 7$

Other notations are sometimes used to illustrate this same type of solution set on a number line. Pictures of solution sets on the number lines are called the graphs of the solution sets or truth sets of the respective mathematical sentences.

Class Exercises

Give the solution set of each of the following sentences, using the set of whole numbers. Then represent each solution set on a number line.

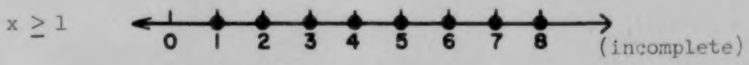
11. $x + 7 = 9$	15. $x + 4 \leq 6$
12. $5y < 7$	16. $x + 4 > 6$
13. $n + 6 < 1$	17. $7z + 5 \leq 30$
14. $3 > 4 - p$	18. $3x - 6 < 5$

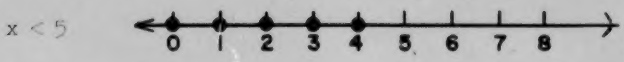
1.5 Compound Number Sentences

The graphing of simple inequalities, as illustrated in the last section, is also a convenient way to find and picture the solution set of compound number sentences. Such sentences are formed by combinations of two or more simple sentences and the connectives "and" and "or." Examples of such compound sentences are given below:

(a) $x \geq 1$ and $x < 5$
(b) $x \leq 1$ or $x > 5$
(c) $(x > 5$ and $x \leq 7)$, or $x < 3$.

Recalling the definition of intersection and union of sets will help us in finding the solution sets for such compound sentences. Example (a) will be true when both simple sentences are true. This means that the solution set we are seeking is the intersection of the solution sets of the two simple sentences considered separately. The two solution sets, again using only the whole numbers, are readily found and graphed.



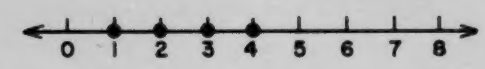


The solution set of $x \geq 1$ is $\{1,2,3,4,\ldots\}$ while the solution set of $x < 5$ is $\{0,1,2,3,4\}$. We may show the solution set of the compound sentence by set notation

$$\{1,2,3,4,\ldots\} \cap \{0,1,2,3,4\} = \{1,2,3,4\}$$

or by a graph:

$x \geq 1$ and $x < 5$



This method of solution by graphing is very useful when the present restriction of whole numbers only is removed. More complicated inequalities such as

$$x^2 - x - 6 \geq 0$$

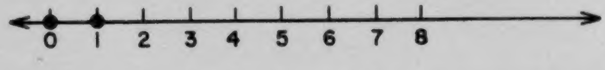
are easily solved by techniques very much like these.

Compound number sentences involving "or" may be solved in a similar way. This time we recall the definition of union of sets and see that the solution of the sentence

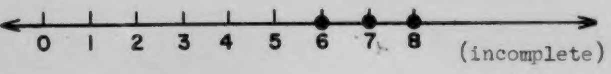
$$x \leq 1 \text{ or } x > 5$$

is the union of the solution sets of $x \leq 1$ and $x > 5$. This is the case since the statement, $x \leq 1$ or $x > 5$, is true when at least one of the two simple sentences is true. Using the graphical method gives us the following:

$x \leq 1$

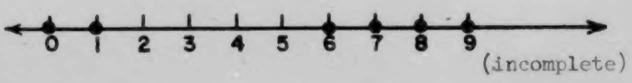


$x > 5$



The solution set of $x \leq 1$ is $\{0,1\}$ and the solution set of $x > 5$ is $\{6,7,8,...\}$. Their union is $\{0,1,6,7,8...\}$. Thus, the graph of the solution set to the compound sentence is

$x \leq 1$ or $x > 5$



When confronted with a compound sentence formed from three or more simple sentences the technique is much the same. As an example, the solution set for the sentence

$$(x > 5 \text{ and } x \leq 7) \text{ or } x < 3$$

may be found by first finding the solution set for the statement within the parentheses and then combining that with the solution set for $x < 3$. The solution set for the compound sentence within the parentheses is $\{6,7\}$. This, when combined with the solution set of $x < 3$, $\{0,1,2\}$, gives $\{0,1,2,6,7\}$.

Step 1:

$x > 5$

Step 2:

$x \leq 7$

Step 3:

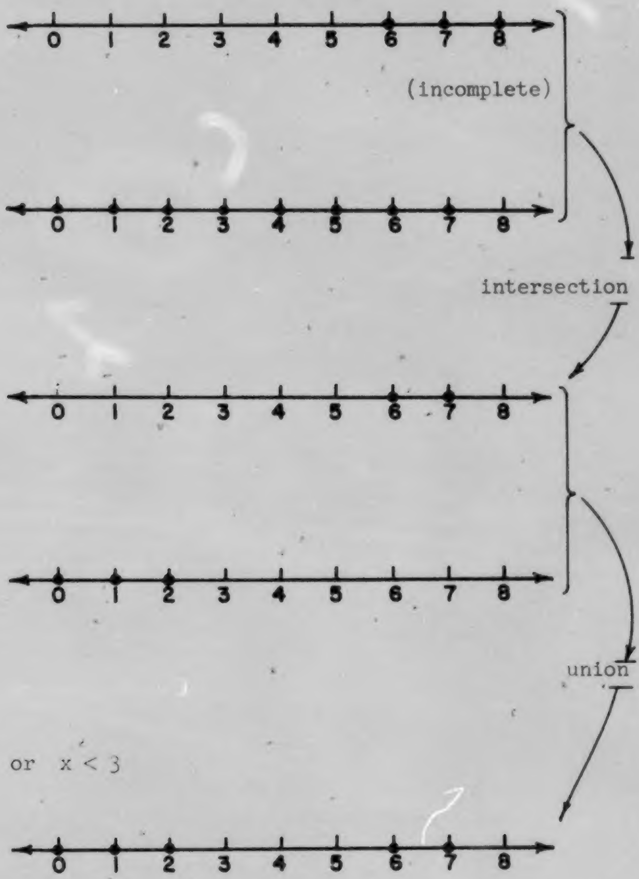
$x > 5$ and $x \leq 7$

Step 4:

$x < 3$

Solution:

$(x > 5$ and $x \leq 7)$ or $x < 3$



Class Exercises

Find the solution set for each of the following sentences using whole numbers. Show each solution on a number line.

19. $x > 5$ and $x < 7$
20. $x > 5$ or $x < 7$
21. $x < 5$ and $x = 7$
22. $x \leq 5$ or $x \geq 7$
23. $x \geq 5$ and $x \neq 7$
24. $(x < 3$ or $x \geq 5)$ and $x \leq 6$
25. $(x < 3$ and $x \geq 5)$ or $x = 5$

1.6 Conclusion

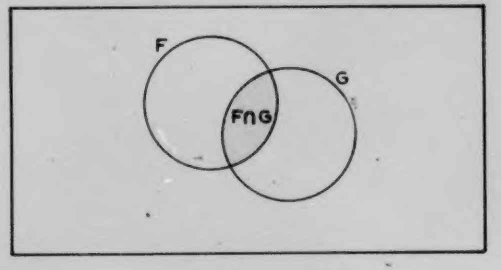
As we stated earlier in this chapter, the language and the properties of sets will be used throughout this book whenever these enable us to expedite our presentation of mathematical ideas about numbers, operations, and geometry.

Let us look again at one of the counting problems from the introduction and analyze it using set language and Venn diagrams. Then we will leave one for you as an exercise. Sometimes it is more important that we find ten ways to do one problem than to find one way to do ten problems!

We will illustrate the first counting problem about the class of 35 pupils, all taking a foreign language. Twenty-one are enrolled in French and 17 in German. How many students are enrolled in both French and German?

Call the pupils taking French set F, and the pupils taking German set G. Then the number of pupils enrolled in both French and German would be the number in the intersection of these two sets: $F \cap G$.

We may draw a diagram of this, the shaded portion being the intersection:



Now, the number of members in set F is 21, and the number of members in set G is 17. Let us denote this by $n(F) = 21$, and $n(G) = 17$.

Now,
$$n(F \cup G) = 35$$
since all 35 pupils are taking at least one foreign language. Also,
$$n(F) + n(G) - n(F \cap G) = 35.$$
Do you agree?

Substituting in the above equation we obtain:
$$21 + 17 - n(F \cap G) = 35;$$
and
$$n(F \cap G) = 3.$$
Therefore, the number in the intersection of the sets, or the number of pupils taking both languages, is 3.

From the diagram it is now a simple matter to find the number of pupils taking only French and the number of pupils taking only German. How many are there in each of these sets?

This is a fairly simple problem and the explanation using set notation may seem longer than the explanation given for the same problem in the introduction. However, with a little experience you will find that more complicated problems of this nature lend themselves easily to illustrations with Venn diagrams and to solutions with set language.

Class Exercises

26. Illustrate the following problem with a Venn diagram and then solve it using set notation.

> In a housing development of 1024 houses, 795 houses have a tree in the front yard and 844 houses have a tree in the back yard. If it is known that every house has at least one tree, how many houses have a tree in both the front and back yard?

Chapter Exercises

1. Write all possible subsets of the set: $\{4,5,6\}$.
2. Draw a Venn diagram to illustrate the following:
   (a) The set whose elements are the Hudson and Ohio Rivers is contained in the set of all rivers in the United States.
   (b) The set whose elements are all tigers, lions, and baboons is contained in the set of all animals.
   (c) The set of numbers 16, 36, and 40 is contained in the set of all counting numbers which are multiples of 4.
3. Given three sets A, B, and C. If $B \subset A$ and $C \subset B$, is $C \subset A$? Illustrate your answer with a Venn diagram.

4. Given the three sets: $A = \{boy, girl, chair\}$, $B = \{girl, chair, dog\}$, and $C = \{chair, dog, cat\}$.
   (a) Find $A \cap B$.
   (b) Show that $A \cap C = C \cap A$.
   (c) Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
   (d) Show that $A \cap (B \cap C) = C \cap (A \cap B)$.

5. (a) Let $\emptyset$ represent the null set, and H any other set. Is the statement $\emptyset \cup H = H \cup \emptyset$ true? Explain your answer.
   (b) Is $\emptyset \cap H = H$? Explain your answer.

6. Let A be the set of even counting numbers; B the set of odd counting numbers; and C the set of all counting numbers.
   (a) Is $A \cup B = C$? Why?
   (b) Is $A \subset C$? Why?
   (c) Is $A \subset B$? Why?
   (d) Is $A \cup B = B \cup A$? Why?
   (e) Is $B \subset A$? Why?
   (f) Draw a Venn Diagram to illustrate $B \subset C$.
   (g) Is $A = B$? Why?

7. (a) Let E be the set of even counting numbers: $\{2, 4, 6, 8, \ldots\}$. Describe a set F so that $E \cup F = C$, when C is the set of all counting numbers.
   (b) What is $E \cap F$?
   (c) Could set F be some other set than what you named in (a)?

8. Given two sets A and B:
   (a) If $A \subset B$, is it true that $A \cup B = B$? Explain your answer.
   (b) If $A \subset B$, is it true that $A \cap B = A$? Explain your answer.

9. Suppose you buy a carton of a dozen eggs. Is it necessary to count the eggs in order to tell whether or not you have a dozen? Why?

10. Draw a Venn diagram to illustrate the problem from the introduction about the 225 students taking foreign languages.

11. Sometimes a many-to-one correspondence between two sets may be defined. If $S = \{1, 2, 3, \ldots, 31\}$ and $T = \{Sun., Mon., Tues., \ldots, Sat.\}$, with the correspondence given by the calendar for July of any one year, show how a many-to-one correspondence may be established.

12. Find the solution set for each of the following sentences using whole numbers. Show each solution on a number line.

    (a) $x > 2$ and $x < 8$.
    (b) $x \geq 9$ and $x < 10$.
    (c) $x \geq 4$ or $x < 4$.
    (d) $x < 9$ or $x > 12$.
    (e) $(x > 5$ and $x < 12)$ or $x \leq 4$

13. Given set W of the whole numbers and set 0 of the odd whole numbers, show with a diagram how a one-to-one correspondence may be set up between these sets. What other observations can you make about the matching of these two infinite sets?

Answers to Class Exercises

1. a. Yes
   b. No
   c. Yes
   d. Yes
   e. Yes

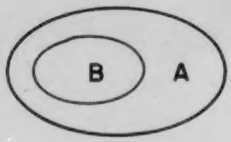
2. Only two examples are given here, but certainly there are others, and answers will vary.
   a. {1,3,5,7,9,11}, or the set of odd whole numbers less than 12.
   b. The set of counting numbers which are multiples of 10 and also less than 101, or the set of multiples of 10 between 0 and 110.
   c. {51,52,53,54,...}, or the set of whole numbers beginning with 51.
   d. $\emptyset$, or the set of boys in your class over 12 feet tall.

3. a. True. 3 is a member of {2,3,4,5}.
   b. False. {0} has the number zero in it whereas $\emptyset$ has no members.
   c. False. {$\emptyset$} has a member, namely, $\emptyset$. $\emptyset$ has no members.
   d. False. {5,6,7,8,...} is an infinite set containing 17.
   e. False. They both include exactly the same members and are therefore equal. The order in which the members are listed is immaterial.
   f. False. The set contains all multiples of 4 less than 100 and thus includes 32.

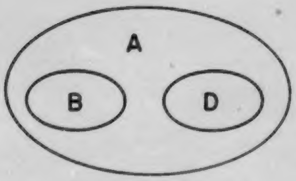
4. a. Infinite.
   b. Finite. It is the empty set.
   c. Finite.
   d. Finite. It is the empty set.
   e. Finite.

5. The elements of set M are 3, 5, 7, 11, 13, 17, 19 and 23. Did you say the odd numbers from 3 through 23 ? Set M was conceived as the set of the first eight odd prime numbers. Moral: One must be very careful when using this notation.

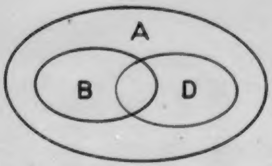
6. a.



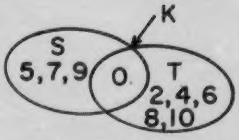
b.



c.



(There are other possibilities)



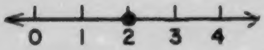
7. a. K = {0}. Yes. Yes.
   b. M = {0, 2, 4, 5, 6, 7, 8, 9, 10}. No. Yes. Yes.
   c. R = {5, 7, 9}. R is a subset of S and M.

8. Eight subsets. Sixteen subsets. 256 subsets. (This takes the form of 2^n where n is the number of members of a set.)

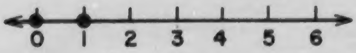
9. a. True
   b. True
   c. False
   d. False
   e. True
   f. True
   g. True
   h. True
   i. True
   j. False
   k. True
   l. False

10. a. ∅
    b. W
    c. W
    d. W
    e. ∅
    f. W

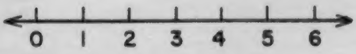
11. {2}



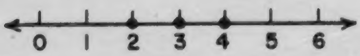
12. {0,1}



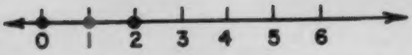
13. ∅



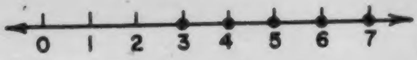
14. {2,3,4}



15. $\{0,1,2\}$

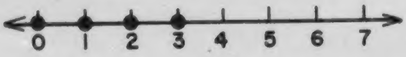


16. $\{3,4,5,6,\ldots\}$

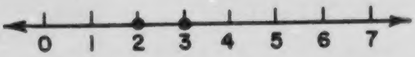


(incomplete)

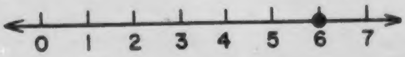
17. $\{0,1,2,3\}$



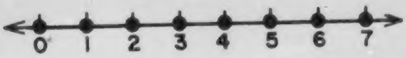
18. $\{2,3\}$



19. $\{6\}$

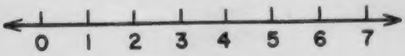


20. $\{0,1,2,3,\ldots\}$

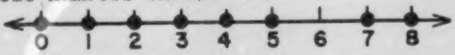


(incomplete)

21. $\emptyset$

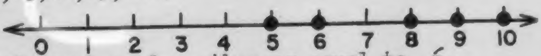


22. The set of all whole numbers except 6.



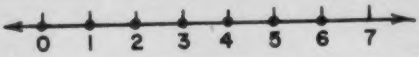
(incomplete)

23. $\{5,6,8,9,10,11,12,13,14,15,\ldots\}$

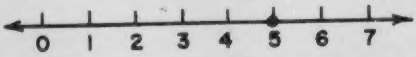


(incomplete)

24. The set of all whole numbers less than or equal to 6.



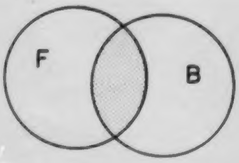
25. $\{5\}$



26. Let H = the set of houses in the development, then $n(H) = 1024$.

25. Let F = the set of houses with a tree in the front yard, then $n(F) = 795$.

Let B = the set of houses with a tree in the back yard, then $n(B) = 844$.



$$\begin{aligned} n(F \cup B) &= 1024 \\ n(F) + n(B) - n(F \cap B) &= 1024 \\ 795 + 844 - n(F \cap B) &= 1024 \\ n(F \cap B) &= 615 \end{aligned}$$

Chapter 2

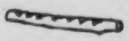
NUMERATION

Introduction

Systems of numeration are invented by men to meet their needs to express number ideas. As civilization has increased these needs, older numeration systems have either expanded or given way to improved systems. Studying the history of the earlier systems provides background for teachers to teach the structure of the decimal system of numeration, deepens pupil's understanding of the principles of numeration, and provides a vehicle for review that is not repetitious.

Through experiences in reading and writing names for large numbers, in using exponents to write new names for numbers, and in using the expanded form for representing numbers, pupils learn that the decimal system may become a powerful vehicle for them. Some familiarity with number bases other than ten makes possible a comparison of these systems with the decimal system, thus reinforcing the understanding of the decimal system. Teachers and pupils who experience "building" a system to a base other than ten no longer take for granted the decimal system of numeration. One additional point should be emphasized here. While the teacher should fully understand the material of this chapter, he must guard against spending too much time in teaching this material to his junior high school students. The understanding of the concept of place value and the relationship of non-ten bases to the decimal system are important. But the students should not be required to memorize the tables of operations or to master completely the skills of computing in different bases. The topic is useful but is not essential for a modern mathematics program.

2.1 Early Numeration Systems

In primitive times men were probably aware of simple numbers in counting, as in counting "one deer" or "two arrows." Their language indicates that they had not learned abstract words for number ideas. Primitive peoples learned to use numbers to keep records. Sometimes they tied knots in a rope, or used a pile of pebbles, or cut marks in sticks to represent the number of objects counted. A boy counting sheep would have • • • • • • • • pebbles, or he might make notches in a stick, as . Each pebble or mark in the stick would represent a single sheep in a one-to-one correspondence between pebbles and sheep. The same kind of record is made when votes in a class election are tallied, as 𝍸 // .

As centuries passed, early people used sounds, or names, for numbers. Today standard sets of names for numbers are used. A rancher counting sheep compares a single sheep with the name "one," and 2 sheep with the name "two" and so on. Man now has symbols (1, 2, 3, ...) and words (eins, zwei, drei,...) which may be used to represent numbers. Word names for numbers vary with the language spoken. For example,

Hindu-Arabic Numerals:	1	2	3
English	one	two	three
German	eins	zwei	drei
Spanish	uno	dos	tres
Latin	unus	duo	tres

A numeration system is a way of naming numbers. It has some basic numerals, and it has ways for making other numerals from them. Different systems of numeration have evolved as their need has arisen.

It is essential that the terms number and numeral be clearly understood. The words are not synonymous. A number is a concept, an idea, an abstraction. A numeral is a symbol, a name for a number. A numeration system is a numeral system, not a number system. It is a system for naming numbers.

Of the ancient systems of numeration, perhaps the Egyptian, Babylonian, Early Roman, Chinese, and Mayan are the best known. A study of Egyptian numerals is given here; it is suggested that other early systems may be introduced by using references, library assignments, class reports, and group projects. Activities of this nature may be correlated with social studies by using the historical period as framework for a system of numeration.

Egyptian System of Numeration

One of the earliest systems of writing numerals for which there is some record is the Egyptian system. Their hieroglyphic, or picture, numerals have been traced as far back as 3300 B.C. Thus, more than 5000 years ago, Egyptians had developed a system with which they could express numbers up to the millions.

The basic Egyptian symbols are shown below:

Our Numeral	Egyptian Symbol	Object Represented
1	|	stroke or vertical staff
10	∩	heel bone
100	𓍢	coiled rope or scroll
1000	𓆼	lotus flower
10,000	𓂭	pointing or bent finger
100,000	𓆐	burbot fish (or polliwog)
1,000,000	𓁨	astonished man

These symbols were carved on wood or stone. The Egyptian system was an improvement over earlier primitive systems because it incorporated the following ideas:

1. A single symbol could be used to represent the number of objects in a collection. For example, the heel bone represented the number ten.
2. The basic symbols could be repeated within a given numeral. For example, the group of symbols 𓍢𓍢𓍢 meant 100 + 100 + 100 or 300.
3. This system was based on groups of ten. Ten strokes are equivalent to a heelbone, ten heelbones are equivalent to a scroll, and so on.

The following table shows how the Egyptians represented certain numerals.

Our Numeral	4	11	23	20,200	1959
Egyptian Numeral	||||	∩|	∩∩|||	𓂭𓂭 𓍢𓍢	𓍢𓍢𓍢 ∩∩ ||| 𓆼 𓍢𓍢𓍢 ∩∩ ||| 𓍢𓍢𓍢 ∩ |||

Note that each basic Egyptian symbol means the same thing regardless of where it is placed. For example, ∩|| , ||∩ , and |∩| all represent the same number, twelve. This is a significant difference from our numeration system where position plays an important role. In our numeration system, the numerals "12" and "21" do not name the same number.

Other Ancient Systems of Numeration

It is likely that most students at this level are familiar with the early Roman system of numeration. If this be the case, then it would be helpful to compare the Roman system to both the Egyptian and the present decimal system of numeration. While based on grouping by tens, Roman numeration also incorporates a modified grouping by fives as illustrated in the table below.

Our Numeral	1	5	10	50	100	500	1000
Roman Numeral	I	V	X	L	C	D	M

In early times the Romans repeated symbols in their numerals the same way that the Egyptians had done many years before. Later, the Romans made use of subtraction to shorten some numerals. Recall that the values of the Roman symbols are added when a symbol representing a larger quantity is placed to the left in the numeral. When a symbol representing a smaller value is written to the left of a symbol representing a larger value, the smaller value is subtracted from the larger. The better student may be interested in exploring on his own some of the other ancient systems of numeration.

Each of the early numeration systems was an improvement over matching objects with notches or pebbles. While it is fairly easy to represent a number in any of the early systems, it is difficult to use the numerals for computing such as in addition and multiplication. It is not as important that students learn to manipulate these numerals as that they learn enough about the systems to compare them with the decimal system of numeration.

Class Exercises

1. How did the Egyptians represent the numbers

   (a) 100; 1000; 10 ?　　(b) 103; 501 ?

2. Write several arrangements of the Egyptian numeral for the number 1,234.

3. How would you add

   (a) ∩∩||| and ∩∩∩|||| ?

   (b) ∩∩∩∩∩∩∩ ||||| and ∩∩∩∩ |||||||| ?

4. Can you devise a plan for multiplication using Egyptian numerals? Try it with ∩∩||| times ||||| .

2.2 Expanded Notation and Exponents

There are many instances in mathematics in which we use a certain number more than once as a factor. Examples are found in the computation of the area of a square, $A = s \times s$; in the volume of a cube, $V = e \times e \times e$; and in the volume of a sphere, $V = \frac{4}{3} \times \pi \times r \times r \times r$.

Another illustration of the use of a number several times as a factor is found in our decimal place value system of numeration. The value represented by a digit in a decimal numeral depends upon the position of the digit in the numeral. Note the different values represented by the two digits "4" in the following example.

$$\begin{aligned} 1484 &= (1 \times 1000) + (4 \times 100) + (8 \times 10) + (4 \times 1) \\ &= (1 \times 10 \times 10 \times 10) + (4 \times 10 \times 10) + (8 \times 10) + (4 \times 1) \end{aligned}$$

Teachers are already familiar with the use of place value in the decimal system as shown in the table on the next page. However, some may not be familiar with writing powers of ten in exponential form. For this reason, an explanation of exponents is given in this section.

Frequently, place values for the decimal system are written more briefly by using the exponential form. In general, the exponent shows how many times the base is used as a factor in a product. Values of the places are read as follows:

1,000,000	10^6	"Ten to the sixth power"
100,000	10^5	"Ten to the fifth power"
10,000	10^4	"Ten to the fourth power"
1,000	10^3	"Ten to the third power"
100	10^2	"Ten to the second power"
10	10^1	"Ten to the first power"
1	10^0	"One"

Numeral	Exponential Form	Place Value	Group Name
1,000,000,000	10^9	10 × 10 × 10 × 10 × 10 × 10 × 10 × 10 × 10	Billion
100,000,000	10^8	10 × 10 × 10 × 10 × 10 × 10 × 10 × 10	Hundred Million
10,000,000	10^7	10 × 10 × 10 × 10 × 10 × 10 × 10	Ten Million
1,000,000	10^6	10 × 10 × 10 × 10 × 10 × 10	Million
100,000	10^5	10 × 10 × 10 × 10 × 10	Hundred Thousand
10,000	10^4	10 × 10 × 10 × 10	Ten Thousand
1,000	10^3	10 × 10 × 10	Thousand
100	10^2	10 × 10	Hundred
10	10^1	10	Ten
1	10^0	1	One

Decimal System Place Value Table

All the numbers represented above are called powers of ten. In 10^6 the 6 is the exponent and the 10 is the base. In 10^1 the 1 is the exponent. Since 10^1 equals 10, the exponent 1 is frequently omitted. However, all other exponents must be written when expressing powers of ten in exponential form.

In 10^0 the 0 is the exponent. Notice that 10^0 has been defined here as being equal to one. In general, we agree to define $a^0 = 1$ for any number a except 0. The convenience of this definition will become apparent later in this section.

The use of exponents enables us to shorten the expanded form of decimal numerals as illustrated below.

$$\begin{aligned} 2603 &= (2 \times 1000) + (6 \times 100) + (0 \times 10) + (3 \times 1) \\ &= (2 \times 10 \times 10 \times 10) + (6 \times 10 \times 10) + (0 \times 10) + (3 \times 1) \\ &= (2 \times 10^3) + (6 \times 10^2) + (0 \times 10^1) + (3 \times 10^0) \end{aligned}$$

The various forms of representing the number 2603 illustrate the use of expanded notation. Writing a numeral in expanded notation explains the meaning of each digit in the numeral. The form using exponents is sometimes simplified, replacing 10^0 by 1 as shown here.

$$\begin{aligned} 32{,}750 &= (3 \times 10^4) + (2 \times 10^3) + (7 \times 10^2) + (5 \times 10^1) + (0 \times 1) \\ &= 30{,}000 + 2000 + 700 + 50 + 0 \end{aligned}$$

Not only are exponents useful in simplifying the writing products; they greatly simplify certain computations. Some examples of computation will indicate the value of using exponents. In each case note the relationship between the exponents of the factors and the exponent of the result.

With 10 as the base:

$$10^1 \times 10^4 = 10 \times (10 \times 10 \times 10 \times 10) = 100{,}000 = 10^5$$

$$10^2 \times 10^2 = (10 \times 10) \times (10 \times 10) = 10{,}000 = 10^4$$

$$10^0 \times 10^3 = 1 \times (10 \times 10 \times 10) = 1{,}000 = 10^3$$

With 2 as the base:

$$2^3 \times 2^2 = (2 \times 2 \times 2) \times (2 \times 2) = 32 = 2^5$$

$$2^2 \times 2^1 = (2 \times 2) \times 2 = 8 = 2^3$$

With 5 as the base:

$$5^2 \times 5^3 = (5 \times 5) \times (5 \times 5 \times 5) = 3125 = 5^5$$

$$5^1 \times 5^3 = 5 \times (5 \times 5 \times 5) = 625 = 5^4$$

Likewise, with a as the base:

$$a^2 \times a^2 = (a \times a) \times (a \times a) = a^4$$

$$a^1 \times a^4 = a \times (a \times a \times a \times a) = a^5$$

$$a^3 \times a^0 = (a \times a \times a) \times 1 = a^3$$

Each of these examples illustrates the property that the product of two powers of a given base can be expressed in exponential form by adding the original exponents. In symbols, this law may be stated as

$$a^m \times a^n = a^{m+n} .$$

Thus, for example, $10^5 \times 10^6 = 10^{5+6} = 10^{11}$. Notice that this law holds when one or both of the original exponents is zero. For example, $10^3 \times 10^0 = 10^{3+0} = 10^3$. Hence, our agreement to define $a^0 = 1$ when $a \neq 0$, makes this law more general.

When it is necessary to use large numbers, working with exponents is especially convenient. For example, the number 7,000 may be expressed as $7 \times 1{,}000 = 7 \times 10^3$; the number $800{,}000 = 8 \times 100{,}000 = 8 \times 10^5$; and the number $63{,}000 = 63 \times 1000 = 63 \times 10^3$. The product $7{,}000 \times 800{,}000$ may be written as: $(7 \times 10^3) \times (8 \times 10^5) = (7 \times 8) \times (10^3 \times 10^5)$.

The earth's weight is approximately 13,000,000,000,000,000,000,000,000 pounds. Now this is a large number to read, to write, or to use in computation. It may be written as 13×10^{24}. Astronomers use ninety-three million miles as the mean distance from the earth to the sun. This number may be expressed by the numeral 93,000,000 , by the product expression $93 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$, or by the exponential form 93×10^6. The exponential form of this number is convenient for most purposes.

Class Exercises

5. Write the following numbers in expanded notation using the exponential form:

   (a) four hundred thirty-six
   (b) five thousand, four hundred nine
   (c) thirty-three thousand, nine hundred eighty-seven
   (d) five million, two hundred fifty-six thousand, eight hundred ninety-eight.

6. Supply the missing parts in the table.

A Decimal Numeral	B Product Expression with Repeated Factors	C Exponential Form	D Powers of Ten
(a) 100	10 × 10	10^2	
(b) 10,000			Fourth
(c)	10 × 10 × 10 × 10 × 10		
(d)			Sixth
(e) 100,000,000			

7. Write in standard form the numeral indicated by:

   (a) (4×10^8)
   (b) $(3 \times 10^6) + (2 \times 10^4) + (5 \times 10^2)$
   (c) $(5 \times 10^5) + (4 \times 10^4) + (3 \times 1)$
   (d) $(6 \times 10^3) + (0 \times 10^2) + (0 \times 10^1) + (6 \times 1)$

8. The earth's weight was given as about 13,000,000,000,000,000,000,000,000 pounds. Express the weight of the earth in exponential form. A pound is approximately equal to 2.2 kilograms. What is the weight of the earth in kilograms?

9. Did you ever hear the name "googol" used for a number? Googol is the name given to a number written as "1" followed by one hundred zeros. Express this number as a power of 10.

2.3 Numeration in Other Bases

There are many familiar activities that utilize the concept of grouping of numbers other than by tens. Questions such as these, chosen from activities of daily living rather than from the context of a mathematics book, may act as a springboard for this section.

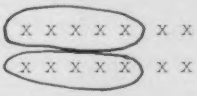
How many eggs are 2 dozen eggs?
How many nickels in 2 quarters?
How many persons sing in a double octet?
How many days are in 2 weeks?
How many persons compose 2 tables of bridge?
How many wheels are on 2 tricycles?
How many shoes are in 2 pairs?
How many trousers in 1 pair?

By investigating other number bases, we become more aware of how our decimal system works.

Base Five

In studying the decimal system of numeration, we grouped sets of objects by tens, and chose names for the place values that were based on powers of ten. The decimal system of numeration is a "base 10" system using ten symbols for building the system.

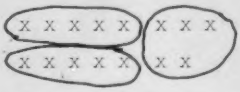
Let us look at a "base 5" system which groups sets of objects by fives. In this system, powers of 5 determine the place values, and five symbols, such as 0, 1, 2, 3, 4 are used for building numerals in the system. Suppose the set of 14 objects represented by x's in the figure below are grouped into sets of fives and ones as shown.



2 sets of fives and
4 sets of one

This grouping may be recorded as 2 fives and 4 ones, or more simply by the numeral 24_{five}. This numeral is read "two four, base five." It is necessary to use the written subscript "five" to designate the base five grouping.

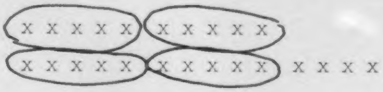
If one more x is added to the set shown above, we would have the following when grouping by fives:



3 sets of fives and
0 sets of one

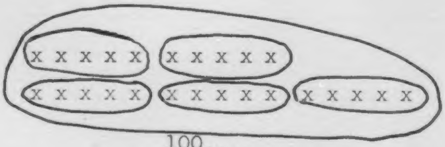
We can represent the number of x's in this set as 30_{five}. This is read as "three zero, base five," and represents the number fifteen.

All grouping in the base five system is by powers of five. Thus, if an x is added to the set shown at the left, we could group by fives to get the figure at the right.



44_{five}

(4 groups of fives
and 4 ones)



100_{five}

(1 group of five × five and
0 fives and 0 ones)

In base five, we rewrite 5 ones as 1 five, 5 fives as 1 twenty-five, and so on. In the figure at the right above, if five is considered a group, then 5 fives may be associated with a group of groups.

Place values in base five numeration are powers of five. Notice how the powers of five are used in expressing 1231_{five} in expanded form.

$$\begin{aligned} 1231_{five} &= (1 \times 125) + (2 \times 25) + (3 \times 5) + (1 \times 1) \\ &= (1 \times 5 \times 5 \times 5) + (2 \times 5 \times 5) + (3 \times 5) + (1 \times 1) \\ &= (1 \times 5^3) + (2 \times 5^2) + (3 \times 5^1) + (1 \times 5^0) \end{aligned}$$

Note that in this example the expanded notations use base ten symbols. One might write 1231_{five} as

$$1_{five} \times (10_{five})^3 + 2_{five} \times (10_{five})^2 + 3_{five} \times (10_{five})^1 + 1_{five} \times (1_{five})$$

in the expanded notation. For psychological reasons we use the notation involving base ten numerals; most of us think quickly and with more feeling in base ten.

But what does the numeral 1231 mean when expressed in other bases?

$$\begin{aligned} 1231_{ten} &= (1 \times 10^3) + (2 \times 10^2) + (3 \times 10^1) + (1 \times 10^0) \\ 1231_{seven} &= (1 \times 7^3) + (2 \times 7^2) + (3 \times 7^1) + (1 \times 7^0) \end{aligned}$$

Normally, when a numeral is written in base ten the subscript word "ten" is omitted. In the remainder of this chapter, if no base is indicated with a numeral, it can be assumed to be in base ten.

Teachers are encouraged to limit the time devoted to the study of number bases other than ten. It is important for pupils to strengthen their background by comparing the structure of such systems with that of the decimal system. Place value and other concepts of structure are learned from the study.

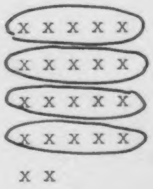
Quarters, nickels, and pennies may be used to illustrate examples in base five using no more than three places (digits). Since 5 pennies equal 1 nickel, and 5 nickels equal 1 quarter grouping of these coins lends itself to grouping in base five.

Class Exercises

10. Look at the example in the diagram.

(a) How many sets of five x's are shown?
How many single x's remain?

(b) Express the number of x's as a base five numeral. Then read the base five numeral.



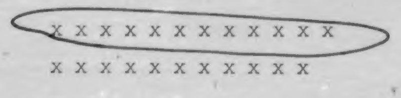
11. Complete the chart:

	Base Five Numeration Sets	Base Five Numeral
x x x x x x x x x x x x x x x x x x x x x x x x	____fives____ones	________
x x x x x x x x x x x x x x x x x	____fives____ones	________
x x x x x x x x x x x x x x x	____fives____ones	________

Base Twelve

In the base ten system of numeration the place values are powers of ten and ten digits are needed. In the base five system, the place values are powers of five and five digits are needed.

Consider next a base twelve system of numeration. It follows that the place values in this system would be powers of twelve and that twelve different digits would be needed. This means that in addition to the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 we must assign two extra digits, say T and E, to represent ten and eleven. To represent twenty-three, the number of x's in the diagram below, as a base twelve numeral, we would need to group by twelves.



Since we get one group of twelve and eleven ones, we would write $1E_{twelve}$. Some other base twelve numerals are listed here. See if you can verify the value of each.

$$\text{sixty-two} = 52_{twelve} = (5 \times 12^1) + (2 \times 1)$$

$$\text{one hundred forty} = E8_{twelve} = (E \times 12^1) + (8 \times 1)$$

$$\text{two hundred sixty-six} = 1T2_{twelve} = (1 \times 12^2) + (T \times 12^1) + (2 \times 1)$$

Although this addition of symbols to the system for writing numerals may seem inconvenient, base twelve has commercial uses. Grouping by twelves lends itself to the business world in activities such as buying eggs by the dozen and pencils by the gross (twelve x twelve).

Base Two

Another familiar system of numeration uses two as a base. Commonly called the binary system it uses only two digits, 0 and 1.

See if you can verify the value of the following binary numerals. Remember, grouping is by two's in the binary system.

$$\text{fifteen} = 1111_{two} = (1 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) + (1 \times 2^0)$$

$$\text{twenty-two} = 10110_{two} = (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) + (0 \times 2^0)$$

$$\text{one hundred forty} = 10001100_{two}$$

Because the binary system uses only two symbols it is particularly adapted to the "on" and "off" switch requirements of modern computers. However, because of the limited number of digits in the system, it is hard to draw a good comparison to the decimal system.

Summary

The following chart is helpful in understanding better the numeral sequence for place value numeration systems with different bases.

Twelve	Ten	Eight	Seven	Five	Four	Three	Two
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	10
3	3	3	3	3	3	10	11
4	4	4	4	4	10	11	100
5	5	5	5	10	11	12	101
6	6	6	6	11	12	20	110
7	7	7	10	12	13	21	111
8	8	10	11	13	20	22	1000
9	9	11	12	14	21	100	1001
T	10	12	13	20	22	101	1010
E	11	13	14	21	23	102	1011
10	12	14	15	22	30	110	1100
11	13	15	16	23	31	111	1101
12	14	16	20	24	32	112	1110
13	15	17	21	30	33	120	1111
14	16	20	22	31	100	121	10000
15	17	21	23	32	101	122	10001
16	18	22	24	33	102	200	10010
17	19	23	25	34	103	201	10011
18	20	24	26	40	110	202	10100

Note that the base numeral always appears as 10 when written in that particular base system. Similarly, the second power of the base (base × base) is indicated by 100 in that base.

It is important that teachers and their students gain an understanding of the structure of numeration systems with different bases, and that they make comparisons among the systems. Memory work is not necessary in this particular study.

Class Exercises

12. Complete the following table.

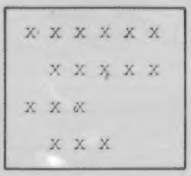
Base	Place Values			
Twelve				
Ten	thousands	hundreds	tens	ones
Eight				
Seven				
Five				
Four	sixty-fours			
Three				
Two				

13. Draw a diagram, using x's, to illustrate each of the numerals given. Circle groups of x's as indicated by the base of each numeral.

(a) 211_{three} (b) 1010_{two} (c) 113_{four}

14. For each part of Exercise 13 write the numeral in expanded notation, using exponents.

15. The diagram represents a set of seventeen objects. In each case complete the sentence given and then represent the number in the corresponding base.



(a) Separate the set into groups of twelve.
There are _______ twelves and _____ ones.

(b) Separate the set into groups of ten.
There are _______ tens and ________ ones.

(c) Separate the set into groups of eight.
There are _______ eights and _______ ones.

(d) Separate the set into groups of seven.
There are _____ sevens and _____ ones.

(e) Separate the set into groups of five.
There are _____ fives and _____ ones.

(f) Separate the set into groups of four.
There are _____ four × fours, _____ fours and _____ ones.

(g) Separate the set into groups of three.
There are _____ three × threes, _____ threes and _____ ones.

(h) Separate the set into groups of two.
There are _____ two × two × two × twos, _____ two × two × twos, _____ two × twos, _____ twos, _____ ones.

2.4 Changing from One Number Base to Another

In Section 2.3 we learned that a particular set of objects may be grouped by tens, by sevens, or by other numbered groups greater than one. This means that we can represent the same number by different numerals when using different bases.

Convert $3E_{\text{twelve}}$ to base ten.

$$\begin{aligned} 3E_{\text{twelve}} &= (3 \times 12^1) + (E \times 1) \\ &= 36 + 11 \\ &= 47 \end{aligned}$$

Convert $23T_{\text{twelve}}$ to base ten.

$$\begin{aligned} 23T_{\text{twelve}} &= (2 \times 12^2) + (3 \times 12^1) + (T \times 1) \\ &= (2 \times 144) + (3 \times 12) + (T \times 1) \\ &= 288 + 36 + 10 \\ &= 334 \end{aligned}$$

Class Exercises

16. By means of expanded notation, convert each numeral to a base ten numeral:

(a) 231_{four}
(b) 35_{seven}
(c) 101011_{two}
(d) 212_{three}
(e) 66_{eight}
(f) 341_{five}
(g) $3T1_{\text{twelve}}$
(h) $T0_{\text{twelve}}$

Changing from Base Ten to Other Bases

You have learned how to change a numeral written in base seven to the corresponding base ten numerals. It is also possible to change from a base ten to a base seven numeral. Let us see how this is done.

In base seven, the values of the places are powers of seven.

$$\begin{aligned} 7^0 &= & &1, \\ 7^1 &= 7 &=\ &7, \\ 7^2 &= 7 \times 7 &=\ &49, \\ 7^3 &= 7 \times 7 \times 7 &=\ &343, \\ 7^4 &= 7 \times 7 \times 7 \times 7 &=\ &2401, \end{aligned}$$

and so on.

Suppose that we wish to change from the base ten decimal numeral 12 to a corresponding base seven numeral. Instead of actually grouping marks, we first think of groups or powers of seven. What is the largest power of seven which is contained in 12 ? Is 7^1 (7) the largest? How about 7^2 (49) or 7^3 (343) ? Only 7^1 and 1 are small enough to be contained in 12. Hence, 7^1 is the largest power of seven included in 12. To find how many sevens are contained in 12, we divide:

$$\begin{array}{r} 1 \\ 7\overline{)12} \\ \underline{7} \\ 5 \end{array}$$

The quotient 1 means that there is 1 seven contained in 12; the remainder 5 means that there are 5 ones left over. Now we are able to write the base ten numeral 12 as a base seven numeral.

$$\begin{aligned} 12 &= (1 \times 7) + (5 \times 1) \\ &= 15_{\text{seven}}. \end{aligned}$$

Consider next the decimal numeral 64. To write this in base seven, we first think of the largest power of seven contained in 64. This is 7^2 or 49. Thus, we can write:

$$64 = (\underline{?} \times 49) + (\underline{?} \times 7) + (\underline{?} \times 1).$$

The first division shown enables us to replace the first blank space with 1. However, the remainder 15 still contains the first power of seven. A second division, this time by 7, gives a remainder of 1. We may now complete the sentence as:

$$\begin{array}{r} 1 \\ 49\overline{)64} \\ \underline{49} \\ 15 \end{array} \qquad \begin{array}{r} 2 \\ 7\overline{)15} \\ \underline{14} \\ 1 \end{array}$$

$$\begin{aligned} 64 &= (1 \times 49) + (2 \times 7) + (1 \times 1) \\ &= (1 \times 7^2) + (2 \times 7^1) + (1 \times 1) \\ &= 121_{\text{seven}}. \end{aligned}$$

Let us change the decimal numeral 524 to a base seven numeral. Recalling again the powers of seven, we select 343 as the largest power contained in 524. Dividing by this power gives a quotient of 1 and a remainder of 181. Next, we divide this remainder by the largest remaining power of seven, 49. We continue to divide each new remainder by decreasing powers of seven. Now we are able to write, in expanded form, the sentence:

$$\begin{array}{rl} & 1 \\ 343 & \overline{)\,524} \\ & \underline{343} \rightarrow (1 \times 343) \\ & 181 \end{array}$$

$$\begin{array}{rl} & 3 \\ 49 & \overline{)\,181} \\ & \underline{147} \rightarrow (3 \times 49) \\ & 34 \end{array}$$

$$\begin{array}{rl} & 4 \\ 7 & \overline{)\,34} \\ & \underline{28} \rightarrow (4 \times 7) \\ & 6 \rightarrow (6 \times 1) \end{array}$$

$$\begin{aligned} 524 &= (1 \times 343) + (3 \times 49) + (4 \times 7) + (6 \times 1) \\ &= (1 \times 7^3) + (3 \times 7^2) + (4 \times 7^1) + (6 \times 1) \\ &= 1346_{\text{seven}}. \end{aligned}$$

In changing base ten numerals to base seven, first select the largest place value of base seven (power of seven) contained in the number. Divide the number by this power of seven and find the quotient and remainder. The quotient is the first digit in the base seven numeral. Divide the remainder by the next smaller power of seven and this quotient is the second digit. Continue to divide each new remainder by each succeedingly smaller power of seven until all the remaining digits in the base seven numeral are found.

In converting base ten numerals to base four numerals, the same procedure is used this time with division being powers of four. As an example, change the decimal numeral 53 to a base four numeral. The powers of four are:

$$4^0 = 1, \quad 4^1 = 4, \quad 4^2 = 16, \quad 4^3 = 64, \quad \text{and so on.}$$

These, of course, become the place values of base four numerals. Successive division by decreasing powers of four give the results shown. Thus, we may write:

$$\begin{array}{rl} & 3 \\ 16 & \overline{)\,53} \\ & \underline{48} \rightarrow (3 \times 16) \\ & 5 \end{array}$$

$$\begin{array}{rl} & 1 \\ 4 & \overline{)\,5} \rightarrow (1 \times 4) \\ & \underline{4} \\ & 1 \rightarrow (1 \times 1) \end{array}$$

$$\begin{aligned} 53 &= (3 \times 16) + (1 \times 4) + (1 \times 1) \\ &= (3 \times 4^2) + (1 \times 4^1) + (1 \times 1) \\ &= 311_{\text{four}}. \end{aligned}$$

As another example,

$$\begin{aligned} 113 &= (1 \times 64) + (3 \times 16) + (0 \times 4) + (1 \times 1) \\ &= (1 \times 4^3) + (3 \times 4^2) + (0 \times 4^1) + (1 \times 1) \\ &= 1301_{\text{four}}. \end{aligned}$$

To convert base ten numerals to base five numerals, study these examples:

$$\begin{aligned} 105 &= (4 \times 25) + (1 \times 5) + (0 \times 1) \\ &= (4 \times 5^2) + (1 \times 5^1) + (0 \times 1) \\ &= 410_{\text{five}}. \end{aligned}$$

$$\begin{aligned} 780 &= (1 \times 625) + (1 \times 125) + (1 \times 25) + (1 \times 5) + (0 \times 1) \\ &= (1 \times 5^4) + (1 \times 5^3) + (1 \times 5^2) + (1 \times 5^1) + (0 \times 1) \\ &= 11110_{\text{five}}. \end{aligned}$$

By divisions such as we have performed in earlier examples, you may verify these conversions.

Thus, by the use of the expanded notation, we are able to do conversions from base ten numerals to numerals in other bases and vice versa.

Class Exercises

17. Change these base ten numerals to base seven numerals:

(a) 509
(b) 31
(c) 28
(d) 350
(e) 2500
(f) 686
(g) 6
(h) 2400

18. Change these base ten numerals to the base indicated:

(a) $85 = \underline{\ \ ?\ \ }_{\text{four}}$
(b) $21 = \underline{\ \ ?\ \ }_{\text{four}}$
(c) $250 = \underline{\ \ ?\ \ }_{\text{four}}$
(d) $250 = \underline{\ \ ?\ \ }_{\text{five}}$
(e) $124 = \underline{\ \ ?\ \ }_{\text{five}}$

2.5 Just For Fun

1. People who work with high speed computers sometimes find it easier to express numbers in the octal, or eight, system rather than the binary system. Conversions from one system to the other can be done very quickly. Can you discover the method used?

2. An inspector of weights and measures carries a set of weights which he uses to check the accuracy of scales. Various weights are placed on a scale to check accuracy in weighing any amount from 1 to 16 ounces. Several checks have to be made, because a scale which accurately measures 5 ounces may, for various reasons, be inaccurate for weighings of 11 ounces and more.

   What is the smallest number of weights the inspector may have in his set, and what must their weights be, to check the accuracy of scales from 1 ounce to 15 ounces? From 1 ounce to 31 ounces?

   This problem is related to the weighing problem posed in the introduction of this book. It is also related to the binary numeration system. Do you see the relationship?

3. (a) Convert the base five numeral 31.42_{five} to a base ten numeral.

   (b) Convert the base ten decimal numeral 22.76 to a base five numeral.

4. Students will enjoy the following card trick which depends upon the use of base two thinking.

A	
1	9
3	11
5	13
7	15

B	
2	10
3	11
6	14
7	15

C	
4	12
5	13
6	14
7	15

D	
8	12
9	13
10	14
11	15

Directions: Make a set of cards as shown. Tell a person to think of a number between 1 and 15 and then to tell you on which card (or cards) it appears.

You can tell him the number by getting the sum of the first number on every card named.

Example: The number 13 is shown on cards A, C, and D. Add 1, 4, and 8 to find the number.

A fuller discussion and extension of this card device may be found on page 41 of the Teacher's Commentary for Junior High School, Volume 1, Part 1.

An interesting discussion and activities on card punching appear in a free booklet, Mathematics in Action, which is obtainable from the Institute of Life Insurance, 277 Park Avenue, New York, N.Y.

5. In the marble problem posed in the introduction of this book, the number of weighings required to locate the heavy marble among a number of marbles was determined. Complete the following table.

Number of Marbles	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Number of Weighings	0	1	1	2	2			2																				

Now write the base three numerals for the numbers from 1 to 28. Do you see a pattern between the number of weighings required as listed in the table and the corresponding base three numerals representing the numbers of marbles weighed? Using this pattern, find how many weighings it would take to do a corresponding problem with 87 marbles.

Chapter Exercises

1. Write the following base ten numbers using
   (1) Egyptian numerals
   (2) Base five numerals
   (3) Base seven numerals

   (a) 19 (b) 53 (c) 666 (d) 1960

2. Write each numeral below in expanded notation, using the exponential form.

   (a) 100_{five}
   (b) 1110_{two}
   (c) 201_{three}
   (d) 110_{four}
   (e) 17_{eight}
   (f) T_{twelve}
   (g) 100_{ten}
   (h) 100_{seven}

3. In the base given, represent one less than each number represented in Exercise 2.

4. Suppose you are paying each amount of money listed in the left column. Rules of the game are (1) that you use only quarters, nickels, and pennies for payment, and (2) that you use the smallest number of coins. Complete the table.

Amount of Money	Number of Quarters	Number of Nickels	Number of Pennies	Base five Numeral
Example: 29 cents	1	0	4	104_{five}
a. 37 cents				
b. 24 cents				
c. 99 cents				
d. 15 cents				
e. 63 cents				
f. $1.24(124 cents)				

5. Make up a base five system using 0, /, ∟, △, and □, for symbols representing the numbers 0, 1, 2, 3, 4. Represent the base ten numbers 1 - 25 in your system.

6. Represent each of the given numerals as a base ten decimal numeral.

(a) 2134_{five} (c) $T5E_{twelve}$

(b) 364_{seven} (d) 20031_{four}

7. Represent each decimal numeral in the base indicated.

(a) $713 = \underline{\ ?\ }_{six}$ (c) $155 = \underline{\ ?\ }_{twelve}$

(b) $44 = \underline{\ ?\ }_{two}$ (d) $333 = \underline{\ ?\ }_{four}$

8. Represent 214_{five} as a base eight numeral.

Answers to Class Exercises

1. (a) 𓍢, 𓆼, ∩ (b) 𓍢|||, 𓍢𓍢 𓍢𓍢 𓍢|

2. 𓆼𓍢𓍢∩∩∩||||, ||𓍢𓆼𓍢∩||∩∩, etc.

3. First, arrange the symbols in the same order:

(a)
∩∩ |||
∩∩∩||||
∩∩∩∩∩ |||||||

(b)
∩∩∩∩∩∩∩ |||||
∩∩∩∩ ||||||||
$\underbrace{\cap\cap\cap\cap\cap\cap\cap\cap\cap\cap}_{\text{Change to 𓍢}}\cap\ \underbrace{||||||||||}_{\text{Change to }\cap}|||$

Answer: 𓍢∩∩ |||

4. It becomes a series of additions.

Add ∩∩||| 5 times.

5. (a) $(4 \times 10^2) + (3 \times 10^1) + (6 \times 1)$

(b) $(5 \times 10^3) + (4 \times 10^2) + (0 + 10^1) + (9 \times 1)$

(c) $(3 \times 10^4) + (3 \times 10^3) + (9 \times 10^2) + (8 \times 10^1) + (7 \times 1)$

(d) $(5 \times 10^6)+(2\times10^5)+(5 \times 10^4)+(6\times10^3)+(3\times10^2)+(9\times10^1)+(8 \times 1)$

6.

A Number Decimal Numeral		B Product Expression with Repeated Factors	C Exponential Form	D Powers of Ten
(a)	100	10 × 10	10^2	Second
(b)	10,000	10×10×10×10	10^4	Fourth
(c)	100,000	10×10×10×10×10	10^5	Fifth
(d)	1,000,000	10×10×10×10×10×10	10^6	Sixth
(e)	100,000,000	10×10×10×10×10×10×10×10	10^8	Eighth

7. (a) 400,000,000 (c) 540,003
(b) 3,020,500 (d) 6,006

8. 13×10^{24}, $2.2 \times 13 \times 10^{24}$.

9. 10^{100}

10. (a) 4 (c) 42_{five}
(b) 2 (d) four, two, base five

11. $\underline{4}$ $\underline{4}$ $\underline{44}_{five}$,

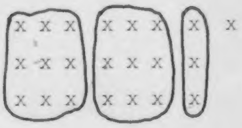
$\underline{3}$ $\underline{2}$ $\underline{32}_{five}$,

$\underline{3}$ $\underline{0}$ $\underline{30}_{five}$.

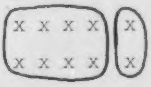
12.

Base	Place Value			
Twelve	One thousand seven hundred twenty-eight	One hundred forty fours	Twelves	Ones
Ten	Thousands	Hundreds	Tens	Ones
Eight	Five hundred twelves	Sixty fours	Eights	Ones
Seven	Three hundred forty - threes	Forty nines	Sevens	Ones
Five	One hundred twenty fives	Twenty fives	Fives	Ones
Four	Sixty-fours	Sixteens	Fours	Ones
Three	Twenty sevens	Nines	Threes	Ones
Two	Eights	Fours	Twos	Ones

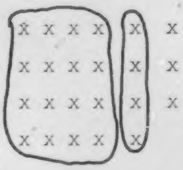
13. (a)



(b)



(c)



14. (a) $211_{three} = (2 \times 3^2) + (1 \times 3^1) + (1 \times 3^0)$

(b) $1010_{two} = (1 \times 2^3) + (0 \times 2^2) + (1 \times 2^1) + (0 \times 2^0)$

(c) $113_{four} = (1 \times 4^2) + (1 \times 4^1) + (3 \times 4^0)$

15. (a) 15_{twelve}

(b) 17_{ten}

(c) 21_{eight}

(d) 23_{seven}

(e) 32_{five}

(f) 101_{four}

(g) 122_{three}

(h) 10001_{two}

16. (a) $$\begin{aligned} 231_{four} &= (2 \times 4^2) + (3 \times 4) + (1 \times 1) \\ &= 32 + 12 + 1 \\ &= 45_{ten} \end{aligned}$$

(b) $$\begin{aligned} 35_{seven} &= (3 \times 7) + (5 \times 1) \\ &= 21 + 5 \\ &= 26_{ten} \end{aligned}$$

(c) $$\begin{aligned} 101011_{two} &= (1 \times 2^5) + (0 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (1 \times 2) + (1 \times 1) \\ &= 32 + 0 + 8 + 0 + 2 + 1 \\ &= 43_{ten} \end{aligned}$$

(d) $$\begin{aligned} 212_{three} &= (2 \times 3^2) + (1 \times 3) + (2 \times 1) \\ &= 18 + 3 + 2 \\ &= 23_{ten} \end{aligned}$$

(e) $$\begin{aligned} 66_{eight} &= (6 \times 8) + (6 \times 1) \\ &= 48 + 6 \\ &= 54_{ten} \end{aligned}$$

(f) $$\begin{aligned} 341_{five} &= (3 \times 5^2) + (4 \times 5) + (1 \times 1) \\ &= 75 + 20 + 1 \\ &= 96_{ten} \end{aligned}$$

(g) $$\begin{aligned} 3T1_{twelve} &= (3 \times 12^2) + (T \times 12) + (1 \times 1) \\ &= 432 + 120 + 1 \\ &= 553_{ten} \end{aligned}$$

(h) $$\begin{aligned} T0_{twelve} &= (T \times 12) + (0 \times 1) \\ &= 120 + 0 \\ &= 120_{seven} \end{aligned}$$

17. (a) $509_{ten} = 1325_{seven}$

$$\begin{array}{r} 1 \\ 343\overline{)509} \\ \underline{343} \\ 166 \end{array} \qquad \begin{array}{r} 3 \\ 49\overline{)166} \\ \underline{147} \\ 19 \end{array} \qquad \begin{array}{r} 2 \\ 7\overline{)19} \\ \underline{14} \\ 5 \end{array}$$

(b) $31_{ten} = 43_{seven}$

$$\begin{array}{r} 4 \\ 7\overline{)31} \\ \underline{28} \\ 3 \end{array}$$

(c) $28_{ten} = 40_{seven}$

(d) $350_{ten} = 1010_{seven}$

$$\begin{array}{r} 1 \\ 343\overline{)350} \\ \underline{343} \\ 7 \end{array} \qquad \begin{array}{r} 0 \\ 49\overline{)7} \\ \underline{0} \\ 7 \end{array} \qquad \begin{array}{r} 1 \\ 7\overline{)7} \\ \underline{7} \\ 0 \end{array} \qquad \begin{array}{r} 0 \\ 7\overline{)0} \\ \underline{0} \\ 0 \end{array}$$

(e) $2500_{ten} = 10201_{seven}$

$$\begin{array}{r} 1 \\ 2401\overline{)2500} \\ \underline{2401} \\ 99 \end{array} \qquad \begin{array}{r} 0 \\ 343\overline{)99} \\ \underline{0} \\ 99 \end{array} \qquad \begin{array}{r} 2 \\ 49\overline{)99} \\ \underline{98} \\ 1 \end{array} \qquad \begin{array}{r} 0 \\ 7\overline{)1} \\ \underline{0} \\ 1 \end{array} \qquad \begin{array}{r} 1 \\ 1\overline{)1} \\ \underline{1} \\ 0 \end{array}$$

(f) $686_{ten} = 2000_{seven}$

$$\begin{array}{r} 2 \\ 343\overline{)686} \\ \underline{686} \\ 0 \end{array} \qquad \begin{array}{r} 0 \\ 49\overline{)0} \\ \underline{0} \\ 0 \end{array} \qquad \begin{array}{r} 0 \\ 7\overline{)0} \\ \underline{0} \\ 0 \end{array} \qquad \begin{array}{r} 0 \\ 1\overline{)0} \\ \underline{0} \\ 0 \end{array}$$

(g) $6_{ten} = 6_{seven}$

(h) $2400_{ten} = 6666_{seven}$

$$\begin{array}{r} 6 \\ 343\overline{)2400} \\ \underline{2058} \\ 342 \end{array} \qquad \begin{array}{r} 6 \\ 49\overline{)342} \\ \underline{294} \\ 48 \end{array} \qquad \begin{array}{r} 6 \\ 7\overline{)48} \\ \underline{42} \\ 6 \end{array}$$

18. (a) $85_{ten} = 1111_{four}$

$$\begin{array}{r} 1 \\ 64\overline{)85} \\ \underline{64} \\ 21 \end{array} \qquad \begin{array}{r} 1 \\ 16\overline{)21} \\ \underline{16} \\ 5 \end{array} \qquad \begin{array}{r} 1 \\ 4\overline{)5} \\ \underline{4} \\ 1 \end{array}$$

(b) $21_{ten} = 111_{four}$

(c) $250_{ten} = 3322_{four}$

$$\begin{array}{r} 3 \\ 64\overline{)250} \\ \underline{192} \\ 58 \end{array} \qquad \begin{array}{r} 3 \\ 16\overline{)58} \\ \underline{48} \\ 10 \end{array} \qquad \begin{array}{r} 2 \\ 4\overline{)10} \\ \underline{8} \\ 2 \end{array}$$

(d) $250_{ten} = 2000_{five}$

$$\begin{array}{r} 2 \\ 125\overline{)250} \\ \underline{250} \\ 0 \end{array}$$

(e) $124_{ten} = 444_{five}$

$$\begin{array}{r} 4 \\ 25\overline{)124} \\ \underline{100} \\ 24 \end{array} \qquad \begin{array}{r} 4 \\ 5\overline{)24} \\ \underline{20} \\ 4 \end{array}$$

Chapter 3

COMPUTATION IN BASES OTHER THAN TEN

Introduction

One purpose of Chapter 2 was to reveal the structure of the decimal system by examining similar structures in other systems; for instance, base five. In this chapter, intended as a sequel to Chapter 2, operations are performed in some of the bases. Such computation has two values for seventh grade students: (1) It provides practice in the fundamental operations free from repetition of the base ten work in elementary school mathematics; and, (2) it strengthens both their skills in and understandings of operations with decimal numerals rather than just supplying more computation.

Emphasis here is placed on base five computation, with suggestions and problems for other bases. The teacher may wish to refer to the base seven study presented in Chapter 2 of the Student's Text, Mathematics for Junior High School, Volume 1, Part 1; and in Studies in Mathematics, Volume VI. Addition and multiplication facts should not be memorized but the process and understanding should be stressed. To this end, once the addition and multiplication tables have been developed, they should be made available to the students for reference when doing computation. Teachers should again be cautioned against spending too much time on work with other number bases. We would hope that poorer students may gain some understanding of the algorithms and good students become proficient in computations. We do not want teachers to drill all students to the point of boredom for better students, and frustration for poorer students.

3.1 Addition

We are going to explore addition in several bases other than ten, with emphasis on base five. Suppose for the moment that we are "moon people" and that we count by fives because there are five fingers on one hand. Students may begin addition in base five by finding sums of simple problems such as:

$3_{\text{five}} + 1_{\text{five}} \qquad (4_{\text{five}})$

$2_{\text{five}} + 0_{\text{five}} \qquad (2_{\text{five}})$

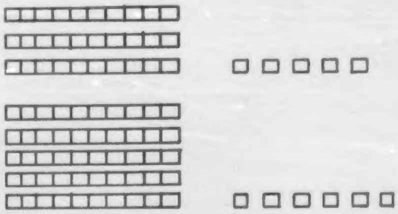
$3_{\text{five}} + 2_{\text{five}} \qquad (10_{\text{five}})$

$2_{\text{five}} + 4_{\text{five}} \qquad (11_{\text{five}})$

Before we can go farther into addition in base five, we shall consider what the addition algorithm really means in base ten. An algorithm is a way of recording the thought processes. In base ten addition let us talk about:

$$\begin{array}{rl} 35 = & 3 \text{ tens} + 5 \text{ ones} \\ \underline{+\ 56} = & \underline{5 \text{ tens} + 6 \text{ ones}} \\ & 8 \text{ tens} + 11 \text{ ones} = 8 \text{ tens} + 1 \text{ ten} + 1 \text{ one} = \\ & \qquad 9 \text{ tens} + 1 \text{ one} = 91. \end{array}$$

To add 35 and 56 it is impractical to draw 35 x's and 56 x's, group them in tens and ones, and then count the number of tens and the number of ones, even though this is what we really mean by addition of whole numbers. To avoid this cumbersome method, we break the problem down into several small problems as indicated in this figure.



Let us combine the small boxes representing ones: 5 + 6 = 11. Combining the larger boxes representing tens, we have: 3 + 5 = 8. Now 11 small boxes is the same as 1 large box and 1 small box. The total, then, is 9 large boxes + 1 small box. This sum is recorded as a written numeral in the addition problem given above.

We may think of any addition problem in this way. In base ten it involves small boxes (1's), large boxes (10's), giant size boxes (100's), economy size boxes (1,000's), family size boxes (10,000's), and so on. Thinking this way in the last problem, we did not need to know the combination 35 + 56; we only needed to know 5 + 6 and 3 + 5. If this line of thought is pursued, one is soon convinced that any addition problem, base ten, may be done if one knows the entries in the table of addition combinations. Likewise, addition in any base can be performed given the table of addition combinations in that base.

Our algorithms exist to eliminate this physical approach to problems. When we perform, mechanically, the addition 35 + 56 = 91, we are indicating this procedure without thinking every step through each time as we did earlier.

Let us now make a base five table of addition. The study of subsequent chapters will reveal several properties which allow us to extend the basic combinations to any number in the system. Making an addition table for base five identifies the twenty-five basic addition combinations to be used in computation.

In teaching a seventh grade class, the addition table for base five can be developed in class. This could be accomplished by preparing the array and inserting only a portion of the entries; students can assist in determining the appropriate entries for the remaining spaces.

Addition Table,Base Five

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

Now let us return to addition in base five. A technique teachers may use with their classes is to demonstrate, with objects such as stars, the following additions in base five.

★★★★ + ★★★ = ★★★★★ + ★★

$4_{five} + 3_{five} = 10_{five} + 2_{five} = 12_{five}$

The concept of grouping by fives is to be emphasized.

If the expanded notation developed earlier is followed in thinking through base five addition, then we have: (using the addition table)

$$\begin{array}{rl}
22_{five} & = 2 \text{ fives} + 2 \text{ ones} \\
+\ 14_{five} & = 1 \text{ five} + 4 \text{ ones} \\
\hline
 & 3 \text{ fives} + 11 \text{ ones} \\
 & = 3 \text{ fives} + (1 \text{ five} + 1 \text{ one}) \\
 & = (3 \text{ fives} + 1 \text{ five}) + 1 \text{ one} \\
 & = 4 \text{ fives} + 1 \text{ one} \\
 & = 41_{five}
\end{array}$$

Observe that the notation "11 ones" is base five notation being used in a base five problem. Here are several more examples expressed in somewhat simplified form.

$$\begin{array}{rl}
21_{\text{five}} & = 2 \text{ fives} + 1 \text{ one} \\
\underline{+\ 24_{\text{five}}} & = \underline{2 \text{ fives} + 4 \text{ ones}} \\
 & \quad 4 \text{ fives} + 10 \text{ ones} \\
 & = (4 \text{ fives} + 1 \text{ five}) + 0 \text{ ones} \\
 & = 10 \text{ fives} + 0 \text{ ones} \\
 & = 1 \text{ five} \times \text{five} + 0 \text{ fives} + 0 \text{ ones} = 100_{\text{five}}.
\end{array}$$

$$\begin{array}{rl}
223_{\text{five}} & = 2 \text{ five} \times \text{fives} + 2 \text{ fives} + 3 \text{ ones} \\
\underline{+\ 243_{\text{five}}} & = \underline{2 \text{ five} \times \text{fives} + 4 \text{ fives} + 3 \text{ ones}} \\
 & \quad 4 \text{ five} \times \text{fives} + 11 \text{ fives} + 11 \text{ ones} \\
 & = 1 \text{ five} \times \text{five} \times \text{five} + 0 \text{ five} \times \text{fives} + 2 \text{ fives} + 1 \text{ one} \\
 & = 1021_{\text{five}}.
\end{array}$$

In each of these cases it has been necessary to "regroup". Regrouping in base five means renaming

10_{five} ones as 1 five
10_{five} fives as 1 five × five
10_{five} five × fives as 1 five × five × five.

This corresponds to the base ten regrouping which means renaming

10 ones as 1 ten
10 tens as 1 ten × ten
10 ten × tens as 1 ten × ten × ten.

In any base regrouping involves an exchange between place value positions as shown; groups correspond to the powers of the base being used. Thus,

10_{five} means 1 group of fives, while
10_{seven} means 1 group of sevens, and
10_{three} means 1 group of threes.

The term regrouping used here is the same process often referred to in base ten as "carrying." When teaching addition in these and other bases it is helpful to have students construct addition tables for easy reference.

Addition Table, Base Seven

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

Addition Table, Base Three

+	0	1	2
0	0	1	2
1	1	2	10
2	2	10	11

A special addition with regrouping occurs with denominate numbers. Tables of measure determine the grouping. Several examples show that regrouping is used with addition of denominate numbers.

$$\begin{array}{rl} & 3421 \text{ feet} \\ + & 4890 \text{ feet} \\ \hline & 8311 \text{ feet} = 1 \text{ mile} + 3031 \text{ feet} \end{array}$$

$$\begin{array}{rrrrr} & 1 \text{ m.} & 8 \text{ dm.} & 3 \text{ cm.} & 5 \text{ mm.} \\ + & 5 \text{ m.} & 5 \text{ dm.} & 8 \text{ cm.} & 3 \text{ mm.} \\ \hline & 6 \text{ m.} & 13 \text{ dm.} & 11 \text{ cm.} & 8 \text{ mm.} \\ = & 7 \text{ m.} & 4 \text{ dm.} & 1 \text{ cm.} & 8 \text{ mm.} \end{array}$$

$$\begin{array}{rrrr} & 4 \text{ grams} & 2 \text{ decigrams} & 8 \text{ centigrams} \\ + & 2 \text{ grams} & 7 \text{ decigrams} & 7 \text{ centigrams} \\ \hline & 6 \text{ grams} & 9 \text{ decigrams} & 15 \text{ centigrams} \\ = & 7 \text{ grams} & 0 \text{ decigrams} & 5 \text{ centigrams} \end{array}$$

$$\begin{array}{rrrr} & 3 \text{ weeks} & 4 \text{ days} & 18 \text{ hours} \\ + & 2 \text{ weeks} & 8 \text{ days} & 6 \text{ hours} \\ \hline & 5 \text{ weeks} & 12 \text{ days} & 24 \text{ hours} \\ = & 6 \text{ weeks} & 6 \text{ days} & 0 \text{ hours} \end{array}$$

Addition in base five or in any other base may be checked by renaming the numerals in decimal notation and adding. For example:

$$\begin{array}{rcr} \underline{\text{Base Five}} & & \underline{\text{Base Ten}} \\ 21_{\text{five}} & \longleftrightarrow & 11 \\ +\ 24_{\text{five}} & \longleftrightarrow & +\ 14 \\ \hline 100_{\text{five}} & \longleftrightarrow & 25_{\text{ten}} \end{array}$$

The addition of two numbers is represented below in four different bases. Verify that each is the same problem simply expressed in a different base from the others.

$$\begin{array}{rcrcrcr}
\underline{\text{Base Ten}} & & \underline{\text{Base Twelve}} & & \underline{\text{Base Eight}} & & \underline{\text{Base Three}} \\
299 & & 20E_{\text{twelve}} & & 453_{\text{eight}} & & 32002_{\text{three}} \\
\underline{+\ 27} & \longleftrightarrow & \underline{+\ 23_{\text{twelve}}} & \longleftrightarrow & \underline{+\ 33_{\text{eight}}} & \longleftrightarrow & \underline{+\ 1000_{\text{three}}} \\
326_{\text{ten}} & & 232_{\text{twelve}} & & 506_{\text{eight}} & & 33002_{\text{three}}
\end{array}$$

Addition in bases other than ten is included in a seventh grade mathematics program because it helps to clarify addition in decimal notation while at the same time illustrating certain number properties. The words "regroup" or "rename" are found in many commercial textbooks; they are preferred by most elementary school teachers over the term "carry," which they replace. An application of regrouping occurs in addition with denominate numbers, as seen in the examples given in this section.

Class Exercises

1. Complete the following table showing the basic addition combinations for base eight.

Addition Table, Base Eight

+	0	1	2	3	4	5	6	7
0								
1								
2								
3								
4								
5								
6								
7								

2. Add the following, noting the base in which each is written.

(a) $\begin{array}{r} 22_{\text{five}} \\ \underline{+\ 13_{\text{five}}} \end{array}$

(b) $\begin{array}{r} 43_{\text{five}} \\ \underline{+\ 14_{\text{five}}} \end{array}$

(c) $\begin{array}{r} 177_{\text{eight}} \\ \underline{+\ 201_{\text{eight}}} \end{array}$

(d) $\begin{array}{r} 321_{\text{eight}} \\ \underline{+\ 275_{\text{eight}}} \end{array}$

3. Check each addition in Exercise 2 by first changing the numerals to base ten.

4. Add as indicated and check using base five numerals.

(a)
$$\begin{array}{r} 75_{ten} \\ +\ 318_{ten} \\ \hline \end{array}$$

(b)
$$\begin{array}{r} 35_{ten} \\ +\ 104_{ten} \\ \hline \end{array}$$

5. Add:

(a)
$$\begin{array}{r} 11_{two} \\ +\ 11_{two} \\ \hline \end{array}$$

(b)
$$\begin{array}{r} 32_{five} \\ 32_{five} \\ 32_{five} \\ 32_{five} \\ +\ 32_{five} \\ \hline \end{array}$$

(c)
$$\begin{array}{r} 43_{five} \\ 43_{five} \\ 43_{five} \\ 43_{five} \\ +\ 43_{five} \\ \hline \end{array}$$

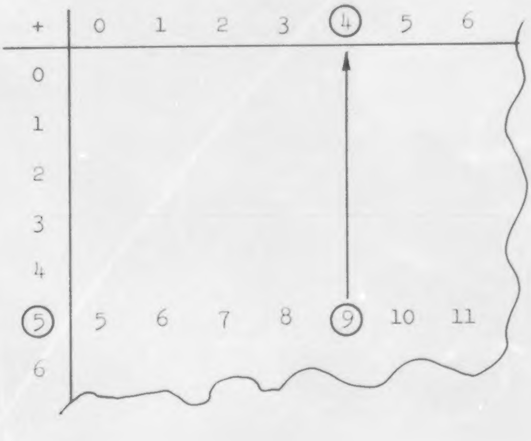
(d)
$$\begin{array}{r} 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ 24 \\ +\ 24 \\ \hline \end{array}$$

(e)
$$\begin{array}{r} 26_{seven} \\ 26_{seven} \\ 26_{seven} \\ 26_{seven} \\ 26_{seven} \\ 26_{seven} \\ +\ 26_{seven} \\ \hline \end{array}$$

Explain the connection between the addends and the sum for each part.

3.2 Subtraction

Most people learn to subtract in base ten by practicing certain subtraction combinations long enough to become thoroughly familiar with them. Suppose we pretend for the moment that you do not know the answer to the subtraction 9 - 5. The answer can be found in the base ten addition table. The question you really need to answer is "What number, when added to 5, yields 9?"



The table shows that addition may be used to solve subtraction problems. Since $4 + 5 = 9$, it is implied that $9 - 5 = 4$. Subtraction is the inverse operation of addition. In a later chapter the concept of inverse operations will be discussed in more detail.

Using only the base five addition table verify that each of these subtraction problems is correct:

$$10_{\text{five}} - 2_{\text{five}} = 3_{\text{five}}$$

$$11_{\text{five}} - 4_{\text{five}} = 2_{\text{five}}$$

$$13_{\text{five}} - 4_{\text{five}} = 4_{\text{five}}$$

A simple subtraction problem in base five, where no regrouping is necessary, is shown below.

$$\begin{array}{rcl} 34_{\text{five}} & = & 3 \text{ fives} + 4 \text{ ones} \\ \underline{-\ 21_{\text{five}}} & = & \underline{2 \text{ fives} + 1 \text{ one}} \\ & & 1 \text{ five} \ + 3 \text{ ones} = 13_{\text{five}} \end{array}$$

More difficult subtraction problems involve regrouping. Just as we reviewed regrouping in addition, let us look at an example of regrouping in base ten subtraction.

$$\begin{array}{rcccl} 55 & = & 5 \text{ tens} + 5 \text{ ones} & = & 4 \text{ tens} + 15 \text{ ones} \\ \underline{-\ 27} & = & \underline{2 \text{ tens} + 7 \text{ ones}} & = & \underline{2 \text{ tens} + \ \ 7 \text{ ones}} \\ & & & & 2 \text{ tens} + \ \ 8 \text{ ones} = 28 \end{array}$$

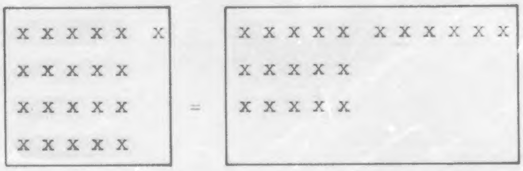
Let us consider a similar subtraction problem using base five numerals. Notice how the logic of the base ten subtraction process using regrouping has been followed.

$$\begin{array}{rcccl} 41_{\text{five}} & = & 4 \text{ fives} + 1 \text{ one} & = & 3 \text{ fives} + 11 \text{ ones} \\ \underline{-\ 23_{\text{five}}} & = & \underline{2 \text{ fives} + 3 \text{ ones}} & = & \underline{2 \text{ fives} + \ \ 3 \text{ ones}} \\ & & & & 1 \text{ five} \ + \ 3 \text{ ones} = 13_{\text{five}} \end{array}$$

The diagram below illustrates how the same problem can be represented by showing regrouping as the partitioning of a set. This method is frequently helpful in explaining the operation of subtraction to students.

Base Five

$$\begin{array}{r} 41_{five} \\ -\ 23_{five} \\ \hline 13_{five} \end{array}$$



Another example of subtraction using base five numerals is shown here. Again note the use of regrouping.

$$\begin{array}{rcrcr} 100_{five} & = & 1\ \text{five} \times \text{five} + 0\ \text{fives} + 0\ \text{ones} & = & 4\ \text{fives} + 10\ \text{ones} \\ -\ 23_{five} & & 2\ \text{fives} + 3\ \text{ones} & = & 2\ \text{fives} + 3\ \text{ones} \\ \hline & & & & 2\ \text{fives} + 2\ \text{ones} = 22_{five} \end{array}$$

Following a similar procedure, teachers may guide students through subtraction problems in other bases. For example:

$$\begin{array}{rcrcr} 21_{three} & = & 2\ \text{threes} + 1\ \text{one} & = & 1\ \text{three} + 11\ \text{ones} \\ -\ 12_{three} & = & 1\ \text{three} + 2\ \text{ones} & = & 1\ \text{three} + 2\ \text{ones} \\ \hline & & & & 2\ \text{ones} = 2_{three}\ . \end{array}$$

Note here that the 2 threes and 1 one were regrouped as 1 three and 11 ones in base three notation.

Subtraction in any base may be checked by changing the numerals to decimal notation and then performing the corresponding subtraction with base ten numerals. As an alternative procedure, teachers may prefer the addition method of checking subtraction problems. This procedure gives more practice in using notation in bases other than ten. Each of the subtraction problems of this section is checked here using the corresponding addition.

Problem:

$$\begin{array}{r} 55 \\ -\ 27 \\ \hline 28 \end{array} \qquad \begin{array}{r} 41_{five} \\ -\ 23_{five} \\ \hline 13_{five} \end{array} \qquad \begin{array}{r} 100_{five} \\ -\ 23_{five} \\ \hline 22_{five} \end{array} \qquad \begin{array}{r} 21_{three} \\ -\ 12_{three} \\ \hline 2_{three} \end{array}$$

Check:

$$\begin{array}{r} 28 \\ +\ 27 \\ \hline 55 \end{array} \qquad \begin{array}{r} 13_{five} \\ +\ 23_{five} \\ \hline 41_{five} \end{array} \qquad \begin{array}{r} 22_{five} \\ +\ 23_{five} \\ \hline 100_{five} \end{array} \qquad \begin{array}{r} 2_{three} \\ +12_{three} \\ \hline 21_{three} \end{array}$$

Just as with addition, the subtraction of denominate numbers provides opportunities for students to develop skill in regrouping numbers used in expressing measurements. This functional aspect of regrouping is related to the study of measurement in Chapter 12. Here is an example of subtraction with denominate numbers:

$$\begin{array}{rrr} 6\text{ m. } 3\text{ cm.} & = \quad 5\text{ m. } 103\text{ cm.} & = \quad 5\text{ m. } 102\text{ cm. } 10\text{ mm.} \\ \underline{-\ 2\text{ m. } 15\text{ cm. } 3\text{ mm.}} & = \quad \underline{-\ 2\text{ m. } 15\text{ cm. } 3\text{ mm.}} & = \quad \underline{-\ 2\text{ m. } 15\text{ cm. } 3\text{ mm.}} \\ & & 3\text{ m. } 87\text{ cm. } 7\text{ mm.} \end{array}$$

Class Exercises

6. Perform the indicated subtractions, noting the base in which each is written.

(a) $\begin{array}{r} 37_{twelve} \\ \underline{-\ 11_{twelve}} \end{array}$ (c) $\begin{array}{r} 42_{seven} \\ \underline{-\ 13_{seven}} \end{array}$ (e) $\begin{array}{r} 44_{eight} \\ \underline{-\ 35_{eight}} \end{array}$

(b) $\begin{array}{r} 122_{three} \\ \underline{-\ 21_{three}} \end{array}$ (d) $\begin{array}{r} 42_{five} \\ \underline{-\ 23_{five}} \end{array}$ (f) $\begin{array}{r} 101_{two} \\ \underline{-\ 10_{two}} \end{array}$

7. Check the subtraction in Exercise 6(c), both

   (a) by verifying the subtraction in base ten, and
   (b) by using addition in the original base.

8. For each pair of numerals, use =, >, or < to make a true statement.

   (a) 30_{five} ______ 2_{seven} (c) 61_{eight} ______ 50_{ten}

   (b) 11_{seven} ______ 20_{four} (d) 1111_{two} ______ 112_{three}

3.3 Multiplication

Multiplication is included in a study of bases other than ten because it reinforces multiplication concepts in base ten, it serves to illustrate certain number properties, and it is a vehicle for multiplication of denominate numbers.

The number of basic multiplication combinations is determined by the base. While in base five there are only 25 basic combinations, base twelve has 144, and base two has 4. Developing the multiplication table for base five identifies the basic multiplication combinations to be used in computation. Teachers may wish to develop the table as a class activity. This may be accomplished by preparing the array and inserting only part of the entries. Students can then obtain appropriate entries for blank spaces. The teacher can offer guidance as indicated in the following discussion. Once completed, the table should be left on the board for easy reference by the students when doing multiplication and division.

Multiplication Table, Base Five

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	11	13
3	0	3	11	14	22
4	0	4	13	22	31

An instructional technique that teachers may use to help students develop these basic multiplication facts for the table relates multiplication to repeated addition. Multiplication of counting numbers is a shortened form of addition in the special case where the addends are equal. For example,

$$3_{\text{five}} \times 4_{\text{five}}$$

may be written as $4_{\text{five}} + 4_{\text{five}} + 4_{\text{five}}$. The multiplication can thus be illustrated by objects in an array composed of three rows of 4 objects each.

x x x x   x x x x
x x x x   x x x x
x x x x   x x x x

$4_{\text{five}} + 4_{\text{five}} + 4_{\text{five}} = 22_{\text{five}}$

The objects are grouped by fives as shown at the right. Both the sum of the addends and the product of the factors is 22_{five}. Notice that the product 22_{five} occurs two places in the multiplication table, for $4_{\text{five}} \times 3_{\text{five}}$ and for $3_{\text{five}} \times 4_{\text{five}}$. This illustrates the commutative property for the multiplication of whole numbers.

Before we consider the use of the table in multiplying with base five numerals, we should look again at multiplication in base ten. In multiplication we not only use the basic multiplication facts, but also a knowledge of the powers of ten. Although you are familiar with the multiplication algorithm, we review it here.

$$\begin{array}{r l}
732_{\text{ten}} & \\
\times\ 7_{\text{ten}} & \\
\hline
14 & \text{—}\ (7 \times 2)\\
210 & \text{—}\ (7 \times 30)\\
4900 & \text{—}\ (7 \times 700)\\
\hline
5124_{\text{ten}} &
\end{array}$$

$$\begin{aligned}
7 \times 732 &= 7 \times (700 + 30 + 2)\\
&= (7 \times 700) + (7 \times 30) + (7 \times 2)\\
&= (4900) + (210) + (14)\\
&= 5124
\end{aligned}$$

In vertical form the multiplication shows partial products and how they are obtained. The multiplication shown in horizontal form uses the distributive property. Because 732_{ten} is a three-place numeral, we recognize that we have three partial products (7×700), (7×30), and (7×2). Taking the sum of the partial products in each case requires renaming.

Now let us examine a similar multiplication using base five numerals. Consider the product:

$$2_{\text{five}} \times 242_{\text{five}}.$$

The vertical form of multiplication again shows the partial products and how they are obtained.

$$\begin{array}{r l}
242_{\text{five}} & \\
\times\ 2_{\text{five}} & \\
\hline
4 & \text{—}\ (2_{\text{five}} \times 2_{\text{five}})\\
130 & \text{—}\ (2_{\text{five}} \times 40_{\text{five}})\\
400 & \text{—}\ (2_{\text{five}} \times 200_{\text{five}})\\
\hline
1034_{\text{five}} &
\end{array}$$

Sometimes it is helpful to the student to express such a problem using expanded notation so that the partial products are listed in horizontal form. This clearly shows how the basic multiplication facts from the table are used.

$$\begin{array}{r l}
242_{\text{five}} = & (2 \times \text{five}^2) + (4 \times \text{five}) + (2 \times \text{one})\\
\times\ 2_{\text{five}} = & \qquad\qquad\qquad\qquad\qquad\qquad (2 \times \text{one})\\
\hline
 & (4 \times \text{five}^2) + (13 \times \text{five}) + (4 \times \text{one})\\
 & = (1 \times \text{five}^3) + (0 \times \text{five}^2) + (3 \times \text{five}) + (4 \times \text{one})\\
 & = 1034_{\text{five}}
\end{array}$$

Multiplication in base five may be checked by changing the numerals to base ten numerals, performing the multiplication, and comparing the two products, as in the following:

$$\begin{array}{rcr} 242_{\text{five}} & \longleftrightarrow & 72 \\ \times\ \ 2_{\text{five}} & \longleftrightarrow & \times 2 \\ \cline{1-1}\cline{3-3} 1034_{\text{five}} & & 144_{\text{ten}}. \end{array}$$

A base system of numeration makes the computation of certain products routine. In any base, the numeral 10 names the base. For example, 10_{ten} names ten, 10_{five} names five, and 10_{twelve} names twelve. Thus, using base ten numerals, 10×10 is the square of the base and is written 100. Using base five numerals, 10×10 is also the square of the base and is written 100. Notice that in any base 100 denotes the square of the base. Similarly, 10×100 in any base denotes the base times the square of the base and is written 1000.

We show a use of this property in the following problem:

$$\begin{array}{rcr} 324_{\text{five}} & = & (3 \times \text{five}^2) + (2 \times \text{five}) + (4 \times \text{one}) \\ \times\ 10_{\text{five}} & & (1 \times \text{five}) + (0 \times \text{one}) \\ \cline{1-1}\cline{3-3} \end{array}$$

Using base five notation throughout this problem can be rewritten as:

$$\begin{array}{rl} 324_{\text{five}} & = (3 \times 100_{\text{five}}) + (2 \times 10_{\text{five}}) + (4 \times 1_{\text{five}}) \\ \times\ 10_{\text{five}} & \qquad\qquad\qquad\qquad\qquad\qquad\quad 10_{\text{five}} \\ \cline{1-1}\cline{2-2} & [3 \times (100_{\text{five}} \times 10_{\text{five}})] + [2 \times (10_{\text{five}} \times 10_{\text{five}})] + [4 \times (1_{\text{five}} \times 10_{\text{five}})] \\ & = [3 \times 1000_{\text{five}}] + [2 \times 100_{\text{five}}] + [4 \times 10_{\text{five}}] \\ & = 3000_{\text{five}} + 200_{\text{five}} + 40_{\text{five}} \\ & = 3240_{\text{five}}\,. \end{array}$$

The same problem written in shortened form appears simply as:

$$\begin{array}{rl} 324_{\text{five}} & \\ \times\ \ 10_{\text{five}} & \\ \cline{1-1} 40 & — (10_{\text{five}} \times 4_{\text{five}}) \\ 200 & — (10_{\text{five}} \times 20_{\text{five}}) \\ 3000 & — (10_{\text{five}} \times 300_{\text{five}}) \\ \cline{1-1} 3240_{\text{five}}. & \end{array}$$

For multiplication in other bases it is desirable first to develop the tables showing the basic multiplication combinations. Base seven and base three tables are given here.

Multiplication Table, Base Seven

×	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	11	13	15
3	0	3	6	12	15	21	24
4	0	4	11	15	22	26	33
5	0	5	13	21	26	34	42
6	0	6	15	24	33	42	51

Multiplication Table, Base Three

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	11

Some examples of multiplication in these bases are shown here.

$$\begin{array}{rl} 563_{\text{seven}} & \\ \times\ \ 5_{\text{seven}} & \\ \hline 21 & \text{—}\ (5_{\text{seven}} \times 3_{\text{seven}}) \\ 420 & \text{—}\ (5_{\text{seven}} \times 60_{\text{seven}}) \\ \underline{3400} & \text{—}\ (5_{\text{seven}} \times 500_{\text{seven}}) \\ 4141_{\text{seven}} & \end{array}$$

$$\begin{array}{rl} 342_{\text{seven}} & \\ \times\ 63_{\text{seven}} & \\ \hline 1356 & \text{—}\ (3_{\text{seven}} \times 342_{\text{seven}}) \\ \underline{30450} & \text{—}\ (60_{\text{seven}} \times 342_{\text{seven}}) \\ 32136_{\text{seven}} & \end{array}$$

$$\begin{array}{rl} 212_{\text{three}} & \\ \times\ 2_{\text{three}} & \\ \hline 11 & \text{—}\ (2_{\text{three}} \times 2_{\text{three}}) \\ 20 & \text{—}\ (2_{\text{three}} \times 10_{\text{three}}) \\ \underline{1100} & \text{—}\ (2_{\text{three}} \times 200_{\text{three}}) \\ 1131_{\text{three}} & \end{array}$$

$$\begin{array}{rl} 121_{\text{three}} & \\ \times\ 12_{\text{three}} & \\ \hline 1012 & \text{—}\ (2_{\text{three}} \times 121_{\text{three}}) \\ 1210 & \text{—}\ (10_{\text{three}} \times 121_{\text{three}}) \\ \hline 2222_{\text{three}} & \end{array}$$

$$\begin{array}{r} 201_{\text{three}} \\ \times\ 20_{\text{three}} \\ \hline 11020_{\text{three}} \end{array}$$

The multiplication at the right may also be expressed using the following development:

$$\begin{aligned} 201_{\text{three}} \times 20_{\text{three}} &= 201_{\text{three}} \times (2_{\text{three}} \times 10_{\text{three}}) \\ &= (201_{\text{three}} \times 2_{\text{three}}) \times 10_{\text{three}} \\ &= 1102_{\text{three}} \times 10_{\text{three}} \\ &= 11020_{\text{three}}. \end{aligned}$$

A special aspect of grouping in multiplication is found in obtaining products where denominate numbers are concerned. For example, "How many weeks and days are in three groups of 3 weeks and 4 days?"

$$\begin{array}{rlrl} & 3 \text{ weeks} + & 4 & \text{days} \\ & & \times & 3 \\ \hline & 9 \text{ weeks} + & 12 & \text{days} \\ = & 10 \text{ weeks} + & 5 & \text{days} \end{array}$$

In this example days are grouped by sevens to make weeks.

Many items are weighed by pounds and ounces. For example, John is mailing 4 packages, each of which weighs 2 pounds and 5 ounces. He needs to know the total weight of the packages.

$$\begin{array}{rrlrl} & 2 & \text{pounds} + & 5 & \text{ounces} \\ & & & \times & 4 \\ \hline & 8 & \text{pounds} + & 20 & \text{ounces} \\ = & 9 & \text{pounds} + & 4 & \text{ounces} \end{array}$$

From linear measurements we find problems such as these:

$$\begin{array}{rrlrlrl} & 4 & \text{yds.} & 2 & \text{ft.} & 7 & \text{in.} \\ & & & & & \times\ 4 & \\ \hline & 16 & \text{yds.} & 8 & \text{ft.} & 28 & \text{in.} \\ = & 19 & \text{yds.} & 1 & \text{ft.} & 4 & \text{in.} \end{array} \qquad \begin{array}{rrlrl} & 2 & \text{m.} & 35 & \text{cm.} \\ & & & \times\ 5 & \\ \hline & 10 & \text{m.} & 175 & \text{cm.} \\ = & 11 & \text{m.} & 75 & \text{cm.} \end{array}$$

Class Exercises

9. Multiply as indicated:

(a) $\begin{array}{r} 31_{five} \\ \times\ 2_{five} \\ \hline \end{array}$ (b) $\begin{array}{r} 42_{five} \\ \times\ 4_{five} \\ \hline \end{array}$ (c) $\begin{array}{r} 234_{five} \\ \times\ 31_{five} \\ \hline \end{array}$

10. Multiply as indicated:

(a) $\begin{array}{r} 300_{seven} \\ \times\ 5_{seven} \\ \hline \end{array}$ (b) $\begin{array}{r} 200_{three} \\ \times\ 2_{three} \\ \hline \end{array}$ (c) $\begin{array}{r} 212_{three} \\ \times\ 21_{three} \\ \hline \end{array}$

11. Check Exercises 9(a) and 10(a) using base ten numerals.

12. (a) What is $10_{five} \times 10_{five}$? $10_{five} \times 100_{five}$?

(b) What is $10_{seven} \times 10_{seven}$? $10_{seven} \times 100_{seven}$?

(c) What is $10_{two} \times 10_{two}$? $10_{two} \times 100_{two}$?

(d) What is the pattern you observe in (a), (b), and (c)?

13. Compute the following, regrouping as necessary:

(a) 4 yds. 2 ft. 5 in.
$\times$ 3

(b) 2 m. 27 cm. 4 mm.
$\times$ 5

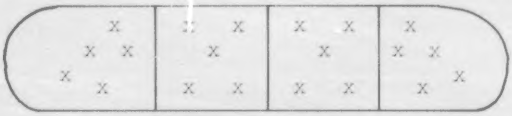
3.4 Division

Teachers may wish to regard the topic of division as optional for some classes, or to treat it along with the study of multiplication. Both discussion and exercises are included here for those who wish to pursue the section. Once again, teachers are urged not to spend an excessive amount of time on this work with other number bases in their seventh grade classes.

For counting numbers we may think of multiplication as repeated addition. As division is to undo multiplication, we may think of division as repeated subtraction. The following example shows continued subtraction of 5 from 20. We may indicate this process by writing $20 \div 5 = 4$, where 4 represents the number of subtractions possible before reaching a remainder less than 5.

$$\begin{array}{r} 20 \\ \underline{-5} \\ 15 \\ \underline{-5} \\ 10 \\ \underline{-5} \\ 5 \\ \underline{-5} \\ 0 \end{array}$$

This approach to division may be shown with a set of 20 objects partitioned into subsets of fives. The model shows 4 subsets, each containing 5 members, thus illustrating $20 \div 5 = 4$.



Perhaps the most useful technique for teaching division relates division to multiplication. As examples, in finding the missing factors in multiplication problems such as

$$4 \times ? = 20 \qquad \text{and} \qquad 5 \times ? = 20,$$

we are actually finding the missing quotients in the corresponding division problems,

$$20 \div 4 = ? \qquad \text{and} \qquad 20 \div 5 = ?$$

Students should be well drilled in the relationship between these two operations. Not only will this help them to see how the multiplication tables in various bases can be used to do division problems, but it will also give them good background for their work in algebra.

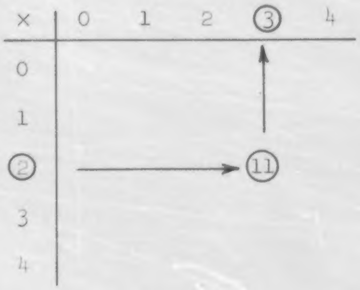
Consider the following division problems using base five numerals:

$$11_{five} \div 2_{five} = ?_{five}$$

The corresponding multiplication problem can be written as:

$$2_{five} \times ?_{five} = 11_{five}$$

2_{five} times what number gives 11_{five}?



The missing factor, 3_{five}, can be found in the base five multiplication table as shown. Thus, we have the quotient:

$$11_{five} \div 2_{five} = 3_{five}.$$

Other simple divisions using base five numerals can be performed in a similar way using the base five multiplication table. Use this method to verify each of the following:

$$14_{five} \div 3_{five} = 3_{five}$$

$$22_{five} \div 4_{five} = 3_{five}$$

$$13_{five} \div 2_{five} = 4_{five}.$$

While the use of physical models is a good learning device, we cannot depend upon drawing models to represent divisions where larger numbers are concerned. Nor can we always find quotients by direct inspection of the multiplication table. Consequently using an algorithm, which is a way of recording or of processing one's thinking, is helpful.

Think of computing $760 \div 20$ with decimal numerals using two forms of the algorithm shown.

$$\begin{array}{rr}
 & \underline{38} \\
 & 8 \\
 & 30 \\
20 & \overline{)760} \\
 & \underline{600} \\
 & 160 \\
 & \underline{160} \\
 & 0
\end{array}
\qquad\qquad
\begin{array}{r|r|r}
20 & 760 & \\
 & \underline{600} & \underline{30} \\
 & 160 & \\
 & \underline{160} & \underline{8} \\
 & 0 & 38
\end{array}$$

Either form of the division records the thinking required to answer the division $760 \div 20 = ?$ which is suggested by the sentence $20 \times ? = 760$.

In terms of partitioning a set, we may consider that:

(1) a set of 760 objects has been partitioned into 20 equivalent subsets, each containing 38 members; or

(2) a set of 760 objects has been partitioned into subsets containing 20 members each, with a total of 38 such subsets.

To use such an algorithm with base five numerals we need to recall the role that the various groupings such as 10_{five}, 100_{five}, and 1000_{five} play as factors in multiplications. For example:

$$23_{five} \times 10_{five} = 230_{five}$$
$$23_{five} \times 100_{five} = 2300_{five}$$
$$23_{five} \times 1000_{five} = 23000_{five}.$$

We can use these facts to do certain divisions such as:

$$230_{five} \div 10_{five} = 23_{five}$$
$$2300_{five} \div 100_{five} = 23_{five}$$
$$23000_{five} \div 23_{five} = 1000_{five}.$$

How do we approach a division such as $233_{five} \div 4_{five}$? Let us use the division algorithm and certain base five multiplication combinations.

$$\begin{array}{r|c|l}
4_{five} & 233_{five} & \\
 & \underline{220} & 30_{five} \text{ ———— } (4_{five} \times 30_{five} = 220_{five}) \\
 & 13 & \\
 & \underline{13} & \underline{2_{five}} \text{ ———— } (4_{five} \times 2_{five} = 13_{five}) \\
 & 0 & 32_{five}
\end{array}$$

Note that the products listed at the right in the illustration show the corresponding multiplications needed. Each, of course, comes from a basic multiplication found in the table.

We can apply a similar procedure for divisions with two-digit divisors.

23_{five}) 32141_{five}	
23000	1000
4141	
2300	100
1341	
1240	30
101	
101	2
0	1132_{five}

32_{five}) 40243_{five}	
32000	1000
3243	
3200	100
43	
32	1
11_{five} (remainder)	1101_{five}

These two divisions may be expressed simply as:

$$32141_{five} \div 23_{five} = 1132_{five}$$

and

$$40243_{five} \div 32_{five} = 1101_{five} \text{ with remainder } 11_{five}.$$

These divisions can be checked using base ten numerals. Teachers may prefer to check division problems by using multiplication in the indicated base. The two division examples in base five are checked by this method.

$$\begin{array}{r} 1132_{five} \\ \times \quad 23_{five} \\ \hline 4001 \\ 2314 \\ \hline 32141_{five} \end{array} \qquad \begin{array}{rl} 1101_{five} & \\ \times \quad 32_{five} & \\ \hline 2202 & \\ 33030 & \\ \hline 40232_{five} & \\ +\ 11_{five} & \text{(remainder)} \\ \hline 40243_{five} & \end{array}$$

It is suggested that division in any base other than ten be approached by constructing the table of basic multiplication combinations for that base. Such an approach strengthens the concept that division is the inverse of multiplication. To have meaning, division in any base must relate to multiplication in that base.

Class Exercises

14. Perform the indicated divisions.

(a) $112_{five} \div 4_{five}$ (c) $302_{five} \div 12_{five}$

(b) $1031_{five} \div 3_{five}$ (d) $1040_{five} \div 23_{five}$

15. Set up a table of multiplication combinations for base three. Perform the indicated divisions and check by multiplication.

(a) $121_{three} \div 2_{three}$ (c) $10010_{three} \div 20_{three}$

(b) $1012_{three} \div 22_{three}$ (d) $1220_{three} \div 12_{three}$

Chapter Exercises

1. Use the array you made in Class Exercise 1 and add each of the following. Check your answers using base ten.

(a) $$\begin{array}{r} 235_{eight} \\ +\ 175_{eight} \\ \hline \end{array}$$ (c) $$\begin{array}{r} 207_{eight} \\ +\ 175_{eight} \\ \hline \end{array}$$ (e) $$\begin{array}{r} 44_{eight} \\ +\ 237_{eight} \\ \hline \end{array}$$

(b) $$\begin{array}{r} 1064_{eight} \\ +\ 253_{eight} \\ \hline \end{array}$$ (d) $$\begin{array}{r} 121_{eight} \\ +\ 567_{eight} \\ \hline \end{array}$$

2. Subtract in base eight, using the numerals in parts (a), (b), and (c) of Exercise 1. Check your subtraction using base ten numerals.

3. Multiply using base eight numerals:

(a) $$\begin{array}{r} 523_{eight} \\ \times\ 5_{eight} \\ \hline \end{array}$$ (b) $$\begin{array}{r} 741_{eight} \\ \times\ 36_{eight} \\ \hline \end{array}$$

4. Divide using base eight numerals:

(a) $4_{eight}\overline{)34_{eight}}$ (b) $13_{eight}\overline{)5146_{eight}}$

5. Write a division sentence suggestedby each of the following products:

(a) Base ten:

$9 \times 8 = 72$

$2 \times 40 = 80$

$5 \times n = 25$

$4 \times n = 24$

$n \times 10 = 100$

(b) Base five:

$2_{five} \times 3_{five} = 11_{five}$

$4_{five} \times n_{five} = 31_{five}$

(c) Base seven:

$4_{seven} \times 5_{seven} = 26_{seven}$

$4_{seven} \times n_{seven} = 33_{seven}$

$n_{seven} \times n_{seven} = 100_{seven}$

6. Find the value of each n (noting the base indicated) in Exercise 3(c).

7. Some completed problems are given below; name the base in which each problem is stated.

(a)
$$\begin{array}{r} 32 \\ +\ 14 \\ \hline 101 \end{array}$$

(b)
$$\begin{array}{r} 63 \\ 63 \\ +\ 63 \\ \hline 252 \end{array}$$

(c)
$$\begin{array}{r} 24 \\ \times\ 4 \\ \hline 132 \end{array}$$

(d)
$$\begin{array}{r} 33 \\ \times\ 14 \\ \hline 242 \\ 33 \\ \hline 1122 \end{array}$$

(e)
$$\begin{array}{r} 13 \\ \times\ 1 \\ \hline 13 \end{array}$$

(f)
$$\begin{array}{r} 24 \\ +\ 24 \\ \hline 48 \end{array}$$

8. In base ten the final digit of an even number is 0, 2, 4, 6, 8.
   (a) For base two numeration, what is the final symbol of an even number?
   (b) Answer the same question for base three.

9. In base ten numeration the final digit of the square of a number is 0, 1, 4, 5, 6, or 9.
   (a) In base two notation what may be said about the final symbol of the square of a number?
   (b) Answer the same question for base three.

10. Is it possible to substitute base ten digits for the letters is

$$\begin{array}{r} \text{SEVEN} \\ +\ \text{EIGHT} \\ \hline \text{ELEVEN} \end{array}$$

so that a correct addition results? A letter always represents the same digit and no digit is represented by more than one letter.

11. Complete the following addition and multiplication problems in base seven:

(a)
$$\begin{array}{r} 264_{\text{seven}} \\ 352_{\text{seven}} \\ +\ ???_{\text{seven}} \\ \hline 1116_{\text{seven}} \end{array}$$

(b)
$$\begin{array}{r} 514_{\text{seven}} \\ \times\ ?\ _{\text{seven}} \\ \hline 2145_{\text{seven}} \end{array}$$

(c)
$$\begin{array}{r} ???_{\text{seven}} \\ \times\ 54_{\text{seven}} \\ \hline 36201_{\text{seven}} \end{array}$$

Answers to Class Exercises

1.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

2. (a) 40_{five} (b) 112_{five} (c) 400_{eight} (d) 616_{eight}

3. (a)

$$\begin{array}{rl} 22_{\text{five}} & = (2 \times 5) + (2 \times 1) = 12 \\ +\ 13_{\text{five}} & = (1 \times 5) + (3 \times 1) = \underline{\ 8} \\ \hline 40_{\text{five}} & = (4 \times 5) + (0 \times 1) = 20 \end{array}$$

(b)

$$\begin{array}{rl} 43_{\text{five}} & = (4 \times 5) + (3 \times 1) \qquad\qquad\qquad = 23 \\ +\ 14_{\text{five}} & = (1 \times 5) + (4 \times 1) \qquad\qquad\qquad = \underline{\ 9} \\ \hline 112_{\text{five}} & = (1 \times 5^2) + (1 \times 5) + (2 \times 1) = 32 \end{array}$$

(c)

$$\begin{array}{rl} 177_{\text{eight}} & = (1 \times 8^2) + (7 \times 8) + (7 \times 1) = 127 \\ +\ 201_{\text{eight}} & = (2 \times 8^2) + (0 \times 8) + (1 \times 1) = \underline{129} \\ \hline 400_{\text{eight}} & = (4 \times 8^2) + (0 \times 8) + (0 \times 1) = 256 \end{array}$$

(d)

$$\begin{array}{rl} 321_{\text{eight}} & = (3 \times 8^2) + (2 \times 8) + (1 \times 1) = 209 \\ +\ 275_{\text{eight}} & = (2 \times 8^2) + (7 \times 8) + (5 \times 1) = \underline{189} \\ \hline 616_{\text{eight}} & = (6 \times 8^2) + (1 \times 8) + (6 \times 1) = 398 \end{array}$$

4. (a)

$$\begin{array}{rcr} 75_{\text{ten}} & = & 300_{\text{five}} \\ +\ 318_{\text{ten}} & = & 2233_{\text{five}} \\ \hline 393_{\text{ten}} & = & 3033_{\text{five}} \end{array}$$

(b)

$$\begin{array}{rcr} 35_{\text{ten}} & = & 120_{\text{five}} \\ +\ 104_{\text{ten}} & = & 404_{\text{five}} \\ \hline 139_{\text{ten}} & = & 1024_{\text{five}} \end{array}$$

5. (a) 110_{two} (b) 320_{five} (c) 430_{five} (d) 240 (e) 260_{seven}

6. (a) 26_{twelve}
(b) 101_{three}
(c) 26_{seven}
(d) 14_{five}
(e) 7_{eight}
(f) 11_{two}

8. (a) $>$ (b) $=$ (c) $<$ (d) $>$

9. (a) 112_{five} (b) 323_{five} (c) 13404_{five}

10. (a) 2100_{seven} (b) 1100_{three} (c) 12222_{three}

12. (a) 100_{five}, 1000_{five}
(b) 100_{seven}, 1000_{seven}
(c) 100_{two}, 1000_{two}
(d) Regardless of the base used, $10 \times 10 = 100$ and $10 \times 100 = 1000$.

13. (a) 14 yd. 1 ft. 3 in. (b) 11 m. 37 cm.

14. (a) 13_{five}
(b) 142_{five}
(c) 21_{five}
(d) 21_{five} + remainder 2_{five}

15.

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	11

(a) 22_{three}
(b) 11_{three}
(c) 112_{three}
(d) 101_{three} (remainder 1_{three})

Chapter 4

MATHEMATICAL SYSTEMS

Introduction

Throughout the work of grades 7, 8, and 9 we are concerned with the development of various sets of numbers together with operations on these numbers. It is just as important that junior high school youngsters see the structure of these systems as that they be able to manipulate the elements of any specific system under some given operation.

In chapters to follow we shall carefully explore various number systems and their properties as they may be developed in the junior high school. However, in this chapter we shall explore several abstract systems in order to illustrate and name some important properties of numbers. This is not the manner in which we recommend that these properties be developed for all seventh grade youngsters! For them we suggest an introduction via a more concrete and familiar situation such as will be explored in the ensuing chapters. The abstract development described here could then well follow later in the year.

Youngsters enjoy working with abstractions, especially after they have completed the mathematics program of grades K-6. Often this helps them to see mathematics in a new light and to see the structure in what may have previously been a jumble of unrelated mechanical procedures.

It is also true, however, that seventh graders may develop the habit of asking why this material need be studied. "Why do I have to know the commutative property?" they will ask. As these various properties are developed in this chapter, examples of their applications are also occasionally given. Such a procedure may well be followed as you present this material to your students. However, you may also have to admit periodically that the "pay-off" for these properties will not come until a later date. The distributive property, for example, is very useful in the study of elementary algebra. Much like a good mystery novel, we are developing the essential features of the plot at this time, but must wait until a later date to unravel the rest of the story.

4.1 Binary Operation

> Given a set of elements, a binary operation is a rule whereby to each pair of elements of the set there corresponds exactly one element.

Seventh graders will be familiar with the concept of a binary operation from their work in arithmetic, although perhaps not with the language. Each of the fundamental operations of arithmetic--addition, subtraction, multiplication, and division--is a binary operation. For example, addition is a binary operation in that this operation assigns exactly one number to any two given numbers. Thus, given 7 and 8, we have $7 + 8 = 15$. Similarly, multiplication is a binary operation; given the numbers 7 and 8, we have $7 \times 8 = 56$.

The term binary is used to emphasize the fact that such operations are only done with two elements at a time. Even in addition we do not add three numbers at once; we add two of them and then to that sum we add the third.

Is subtraction a binary operation? Although $7 - 3 = 4$, we should note that the student who has had no experience with negative numbers is not able to find a number to correspond to $3 - 7$. However, there is a definite number that corresponds to $3 - 7$ and we do, therefore, consider subtraction to be a binary operation. Some students will recognize that $3 - 7 = {}^{-}4$, whereas others will learn this fact in a later course.

(It is interesting to note that not all mathematicians will agree on this point. Some will argue that the operation is not a binary one unless it produces a result which is itself a member of the original set of elements.)

An interesting way to introduce youngsters to this concept of an operation is to have them "discover" the meaning of certain abstract operations. Consider, for example, a binary operation symbolized by $*$. The symbol $*$ is a sign of the operation. Thus, $2 * 3$ tells you to operate on 2 and 3 in a certain way. Following are several illustrations of the use of this operation. See if you can discover the meaning of $*$.

$$2 * 3 = 6 \qquad\qquad 4 * 8 = 13$$
$$3 * 5 = 9 \qquad\qquad 5 * 2 = 8$$

In this case the binary operation $*$ tells you to add one to the sum of the two given numbers. That is, for any numbers a and b:

$$a * b = (a + b) + 1.$$

Note that you also obtain the same result by defining the operation $*$ in either of the following ways:

$$a * b = (a + 1) + b$$
$$a * b = a + (b + 1)$$

An interesting classroom activity is to have youngsters invent their own binary operation, present examples of its use, and allow other members of the class to discover its meaning. You will need to set up some sort of guidelines here or this activity can soon get out of hand. For example, it will be almost impossible for a class to discover the meaning of a binary operation that means you are to increase the first number by 5, decrease the second number by 3, and then find their product! Be sure to keep the meaning of the operation within reason. The class exercises below are examples of this type of activity.

Class Exercises

Use the examples given to discover the meaning of the operations $\bigcirc$, $\perp$, $\triangle$, $\sim$, $\sqcup$.

1. $2 \bigcirc 5 = 6$
   $3 \bigcirc 3 = 5$
   $4 \bigcirc 7 = 10$
   $1 \bigcirc 2 = 2$

2. $3 \perp 5 = 4$
   $6 \perp 8 = 7$
   $2 \perp 10 = 6$
   $4 \perp 12 = 8$

3. $3 \triangle 5 = 5$
   $7 \triangle 1 = 7$
   $6 \triangle 9 = 9$
   $8 \triangle 8 = 8$

4. $3 \sim 4 = 10$
   $1 \sim 3 = 5$
   $3 \sim 0 = 6$
   $5 \sim 1 = 11$

5. $3 \sqcup 4 = 5$
   $1 \sqcup 2 = 9$
   $2 \sqcup 8 = 2$
   $7 \sqcup 5 = 0$

4.2 A Mathematical System

A mathematical system is a set of elements with one or more binary operations defined on the set. The elements do not have to be numbers, although they most often are as they are encountered in a seventh grade mathematics class. It may be interesting first to explore the properties of an abstract system where the elements are not numbers.

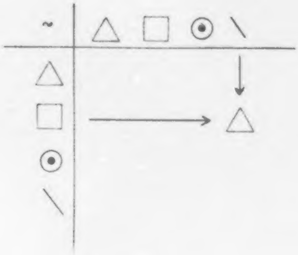
Let us consider a set, M, of elements:

$$M = \{\triangle, \square, \odot, \diagdown\}.$$

Also consider a binary operation, ~, that combines any two members of set M. We can define this operation by means of the following table:

~	△	□	⊙	╲
△	△	□	⊙	╲
□	□	⊙	╲	△
⊙	⊙	╲	△	□
╲	╲	△	□	⊙

This table is read by locating the first of two given elements in the row headings to the left of the table. The second element is then located in the column headings at the top of the table. The result is found within the table where the row and column intersect. For example, to find □ ~ ╲ , we first look



in the row headings at the left of the table until we find the first symbol, □, and then move to the right to the column headed by the second element, ╲ , to find the result, △. Thus, □ ~ ╲ = △. In a similar manner, verify that each of the following is correct:

╲ ~ ⊙ = □
△ ~ ⊙ = ⊙
⊙ ~ □ = ╲

We now have a mathematical system consisting of the set M and the binary operation ~. Note that it really does not matter what the operation ~ means; the operation is defined by the table and we learn about it by comparing the table to discover the properties. Here it is worthwhile to let students examine the table and attempt to discover some properties on their own. What can they discover? What can you discover?

For one thing, all of the entries in the table are members of set M; no new symbol appears. We describe this property by saying that the set M is closed with respect to the operation ~ . This is the *closure property*. In general:

> A set S is said to be *closed* under
> a binary operation * if for any
> elements x and y of S, x * y
> is an element of S.

Thus, the set of whole numbers is closed under addition because the sum of any two whole numbers is a whole number. Do you see that the set of whole numbers is *not* closed under subtraction?

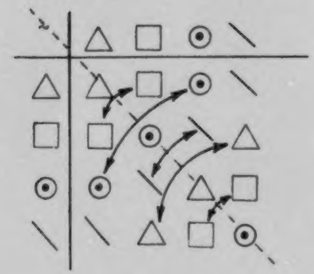
Let us see what else we can discover from the table given at the beginning of this section. Compare your answers to parts (a) and (b) of each of the following examples.

1. (a) □ ~ ⊙ = ?   2. (a) \ ~ □ = ?   3. (a) △ ~ ⊙ = ?
(b) ⊙ ~ □ = ?   (b) □ ~ \ = ?   (b) ⊙ ~ △ = ?

Do you see that the answers are the same in each pair? Indeed this will be true for any pair of elements selected from this table, as you can verify by examination. We say that the operation ~ is *commutative*. By this we mean that the result is independent of the *order* in which the operation is performed. Not all operations are commutative. In general:

> An operation * defined on a set S
> is said to be *commutative* if for any
> elements x and y of S:
>
> x * y = y * x.

There is an easy way to discover whether or not an operation is commutative if one has a table that defines the operations. If there is symmetry with respect to a diagonal line drawn from the upper left to the lower right corner, then we have commutativity.



Note that each element on one side of the dotted line is symmetric to and corresponds to a like element of the other side of the line. Results that come from combining two elements in different order always occur in these corresponding positions in the table. Hence, if reordering the elements operated on does not change the result, these corresponding entries must always agree.

Of course it is not always feasible nor possible to construct a table that defines an operation. For example, if our original set of elements were to consist of the set of real numbers, we would have an infinite collection that could not be accommodated in a table. In such a case the diagonal line test for commutativity could not be used.

Let's consider one more property in this section, this time involving three elements. Since a binary operation relates only two members of a set, we need to make use of parentheses in order to determine which pair of elements to combine first. Without parentheses, the operation might be ambiguous. For example, consider the problem

$$12 \div 6 \div 2.$$

If one divides from left to right the result is $(12 \div 6) = 2$ and $2 \div 2 = 1$. On the other hand, if one divides 6 by 2 first, the result is $12 \div (6 \div 2) = 12 \div 3 = 4$. Thus, the problem, as originally stated, is ambiguous unless parentheses are used or some agreement is made concerning the order in which the elements are grouped for the binary operations. On the other hand, a statement such as $12 + 6 + 2$ is not ambiguous since $(12+6)+2 = 18+2 = 20$ and also $12 + (6 + 2) = 12 + 8 = 20$. Here the grouping does not affect the result. Of course, addition still remains a binary operation; only two elements are added at a time. The point is that in addition, the way the elements are grouped does not affect the result, whereas in division it does.

Now let us evaluate an expression involving three elements of set M keeping in mind that operations within parentheses are to be done first.

$$\begin{aligned}(\odot \sim \square) \sim \triangle &= \diagdown \sim \triangle \\ &= \diagdown\end{aligned}$$

Here the same three elements are grouped in a different way:

$$\begin{aligned}\odot \sim (\square \sim \triangle) &= \odot \sim \square \\ &= \diagdown\end{aligned}$$

Note that the result is the same in each case. That is,

$$(\odot \sim \square) \sim \triangle = \odot \sim (\square \sim \triangle).$$

Will this be true for all arrangements of three elements from set M ? Evaluate the following:

1. (a) $(\diagdown \sim \odot) \sim \square = ?$ 2. (a) $(\square \sim \diagdown) \sim \odot = ?$
   (b) $\diagdown \sim (\odot \sim \square) = ?$ (b) $\square \sim (\diagdown \sim \odot) = ?$

You should find that the answers for each pair of examples are the same and this will be true for any similar arrangement of three elements from set M. We say that the operation ~ is associative. In general:

> An operation * defined on a set S
> is said to be associative if for any
> elements x, y, and z of S:
>
> $$(x * y) * z = x * (y * z).$$

You should note that it is not possible to tell whether an operation is associative by looking at a table. Nor may one assume associativity on the basis of several examples only. Actually every combination of three elements would have to be tried in order to prove associativity. On the other hand, if just one example can be found when the property does not hold, then the operation is not associative. Likewise, if the mathematical system is not closed, then the associative property cannot hold.

Class Exercises

Use the following information for Exercises 6-13:

$A = \{a, b, c\}$

$\otimes$	a	b	c
a	b	c	a
b	c	b	a
c	a	c	b

$L = \{1, 3, 5, 7\}$

$\oplus$	1	3	5	7
1	3	7	1	5
3	5	7	1	3
5	3	1	5	7
7	7	5	3	1

6. $a \otimes c = ?$

7. $3 \oplus 7 = ?$

8. $a \otimes (b \otimes c) = ?$

9. $(7 \oplus 3) \oplus 1 = ?$

10. Is the set A closed with respect to $\otimes$?

11. Is the set L closed with respect to $\oplus$?

12. Does $3 \oplus 5 = 5 \oplus 3$? Is the operation $\oplus$ a commutative one? (Try $7 \oplus 3$ and $3 \oplus 7$.)

13. Is the operation $\otimes$ a commutative operation?

For each of the following described sets and operations determine whether the set is closed. Find which operations are commutative and which are associative.

14. Set: All counting numbers between 25 and 75.
    Operation: Choose the smaller number.
    Example: 28 combined with 36 produces 28.

15. Set: All even numbers between 39 and 61.
    Operation: Choose the first number.
    Example: 52 combined with 46 produces 52;
    46 combined with 52 produces 46.

4.3 Mathematical Systems - Additional Properties

Let us return to set M of the previous section and discover several additional properties of the mathematical system developed there. For convenience, here again is the table defining the operation ~ :

	△	□	⊙	╲
△	△	□	⊙	╲
□	□	⊙	╲	△
⊙	⊙	╲	△	□
╲	╲	△	□	⊙

Now note the following:

□ ~ △ = □   △ ~ □ = □

⊙ ~ △ = ⊙   △ ~ ⊙ = ⊙

╲ ~ △ = ╲   △ ~ ╲ = ╲

△ ~ △ = △

Do you see that the combination of any element of M with △ produces the original member of M ? In other words, the element △ plays the same role here as 0 plays in addition. Recall that the sum of any number and zero is that number. Thus, for any number n,

$$n + 0 = 0 + n = n.$$

The number 1 plays a corresponding role with respect to multiplication. For any number n,

$$n \cdot 1 = 1 \cdot n = n.$$

Likewise, the element △ plays the same role with the operation ~ .

$$n \sim \triangle = \triangle \sim n = n.$$

We call such elements identity elements. An identity element does not change the identity of any element with which it is combined through the operation.

In general:

> An element I is said to be an identity element for the operation $*$, defined on a set S, if $x * I = I * x = x$ for each element x of S.

Note that 0 is the identity element for addition, and that 1 is the identity element for multiplication. Is there an identity element for subtraction or for division? Explain your answer.

As another example consider the following table for an operation which we might call ★. First confirm that the operation ★ is a commutative one.

★	A	B	C	D
A	B	C	D	A
B	C	D	A	B
C	D	A	B	C
D	A	B	C	D

Is there an identity element for ★ ? Could it be A ? If so, then how must the element A behave with respect to the operation ★ ? Is $B \bigstar A = A \bigstar B = B$? We see, from the table, that the answer to the question is "no" and thus, A cannot be an identity element. Neither can B be an identity element since $A \bigstar B$ is not A. However, D is an identity for ★, since

$$\begin{aligned} A \bigstar D = D \bigstar A &= A, \\ B \bigstar D = D \bigstar B &= B, \\ C \bigstar D = D \bigstar C &= C, \\ D \bigstar D &= D. \end{aligned}$$

In the table compare the column under D with the column under the ★. Compare the row to the right of D with the row to the right of the ★. What do you notice? Does this suggest a way to look for an identity element when you are given a table for the operation?

Whenever we have an identity element for an operation, it may be that we also have what are called inverse elements. When the operation is multiplication for real numbers, the identity element is 1. If the product of two numbers, a and b, is the multiplicative identity, 1, then we call each of the numbers the multiplicative inverse of the other. Thus, the multiplicative inverse of 3 is $\frac{1}{3}$ since $3 \times \frac{1}{3} = 1$. You may recognize the multiplicative inverse of a number as its reciprocal.

Suppose the operation is addition. Here 0 is the identity element and we call two numbers additive inverses if their sum is 0; that is, combining the two numbers by addition gives 0. For example, the additive inverse of 3 is $^{-}3$ since $3 + (^{-}3) = 0$. We say the additive inverse of a number is its opposite.

Let us return to the set M described earlier in this section. Recall that we found that the set contained an identity element, namely △. Now let us see if each element has an inverse with respect to the operation ~ . For example, to determine the inverse of □ we must find some element to combine with □ that will produce the identity △. We find this to be ＼ since □~＼ = △. Similarly, the inverse of ＼ is □ since ＼ ~□ = △. The inverse of △ is △ and the inverse of ⊙ is ⊙:

$$\triangle \sim \triangle = \triangle$$
$$\odot \sim \odot = \triangle$$

In general:

> Two elements x and y of set S are said to be inverses of each other under a binary operation * if
>
> $$x * y = y * x = I$$
>
> where I is the identity element for the given set S.

It is possible that within a given set only certain elements have inverses. However, it is impossible for any elements to have inverses if there is no identity element.

The concepts of identity and inverse elements are important ones in the development of mathematics. Let's pause to consider several illustrations of their use in elementary mathematics. Additional illustrations will be given in later chapters.

The identity element for multiplication, 1, is useful in explaining the principle involved in simplifying fractions. For example:

$$\begin{aligned}\frac{9}{12} &= \frac{3}{4} \times \frac{3}{3}\\ &= \frac{3}{4} \times 1\\ &= \frac{3}{4}\end{aligned}$$

A similar procedure is used in simplifying algebraic fractions;

$$\begin{aligned} \frac{2x-4}{3x-6} &= \frac{2(x-2)}{3(x-2)} && \text{by factoring} \\ & && (x \neq 2) \\ &= \frac{2}{3} \cdot \frac{x-2}{x-2} \\ &= \frac{2}{3} \cdot 1 \\ &= \frac{2}{3} \end{aligned}$$

We make use of both of the concepts developed in this section in solving simple equations. Below is a detailed exploration of the solution for the equation $2x + 3 = 7$. (Of course, we normally do not go through each of these steps in this formal a manner.) Note the use of identity elements and inverses of elements in this development.

$$\begin{aligned} 2x + 3 &= 7 \\ (2x + 3) + ({}^-3) &= 7 + ({}^-3) && \text{(The additive inverse of } 3 \text{ is } {}^-3\text{)} \\ 2x + (3 + {}^-3) &= 7 + ({}^-3) && \text{(By the associative property for addition)} \\ 2x + 0 &= 4 && \text{(Here } 0 \text{ is the additive identity)} \\ 2x &= 4 \\ \tfrac{1}{2}(2x) &= \tfrac{1}{2}(4) && \text{(The multiplicative inverse of } 2 \text{ is } \tfrac{1}{2}\text{)} \\ (\tfrac{1}{2} \cdot 2) \cdot x &= \tfrac{1}{2}(4) && \text{(By the associative property for multiplication)} \\ 1 \cdot x &= 2 && \text{(Here } 1 \text{ is the multiplicative identity)} \\ \text{Thus:} \quad x &= 2 \end{aligned}$$

Class Exercises

Use the accompanying table to answer the following questions relative to set $K = \{a,b,c,d,e\}$ and operation $*$ given in the table.

*	a	b	c	d	e
a	d	e	a	b	c
b	e	a	b	c	d
c	a	b	c	d	e
d	b	c	d	e	a
e	c	d	e	a	b

16. $b * c = ?$
17. $(a * a) * d = ?$
18. $e * (d * e) = ?$
19. Is the set K closed with respect to the operation $*$?
20. Is $*$ a commutative operation?

21. Does the set K contain an identity element with respect to * ? If so, what is it?

22. Name the inverse of each of the elements of set K.

4.4 Clock Arithmetic

The numerous properties that we have explored thus far are all important ones with which seventh graders must become familiar. Of course, they should become acquainted with the concepts, if not the actual language, long before they ever enter the seventh grade.

It is frequently difficult to convince junior high school youngsters of the importance of these concepts if they are explained only in terms of ordinary arithmetic. They see little to get excited about in the fact that $2 + 3 = 3 + 2$, or that $5 + 0 = 5$. It is for this reason that it is often advisable to present these ideas in unique settings where possible. Of course, this depends upon the background and ability of the group in question.

One interesting mathematical system that junior high school youngsters enjoy exploring is clock arithmetic. On our twelve-hour clock it is perfectly reasonable to say that $8 + 7 = 3$. (That is, seven hours after 8 o'clock it will be 3 o'clock.) Now we can set up an entire mathematical system based on addition on the twelve-hour clock. Youngsters can be encouraged to complete an addition table for this system and to explore its properties.

Similar activities can be developed around the twenty-four hour clock used in the navy where $9 + 11 = 20$ and $23 + 5 = 4$.

Class Exercises

Consider the elements involved in a twelve-hour clock together with the operation of addition.

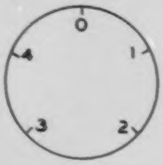
23. $9 + 10 = ?$

24. $11 + 11 = ?$

25. Does this system have an identity element? If so, what is it?

26. What is the inverse of 5 with respect to addition? Explain your answer.

Let us finally turn our attention to arithmetic on a clock with fewer positions. Consider one with a set of five symbols: {0,1,2,3,4}.



We may introduce a binary operation on these symbols by considering clockwise rotations. This definition will be in terms of adding or combining one clockwise rotation with another clockwise rotation. For this reason we call the operation addition and use the symbol +.

We shall consider the starting point to be the position labeled 0. (This is standard practice, although we could have used 5 or any other symbol as well.) For example, 3 + 4 means that we start at 0 and move to position 3. From this position we move 4 more steps to arrive at position 2. Thus, 3 + 4 produces 2. We might simply write this as 3 + 4 = 2. However, this type of arithmetic is an example of modular arithmetic where such an addition is usually written

$$3 + 4 \equiv 2 \pmod 5$$

and is read:

"Three plus four is equivalent to two, modulo 5."

The word "mod" stands for modulus or modulo.

It is customary in the theory of numbers to treat two numbers as equivalent in a given modulus if they have the same remainders when divided by the modulus. For example, $7 \equiv 2 \pmod 5$ since both 7 and 2 give a remainder of 2 when divided by 5. However, in the application of this idea to our clock arithmetic, we restrict ourselves only to the elements in the finite set {0,1,2,3,4}. Examples and discussions of this application may be found in many junior high books. This approach through clock arithmetic is a simple way to start such discussions, and, of course, provides opportunities to examine again some of the basic properties we are developing.

In this example the modulus is 5 which means that there are five positions on the face of the clock. The symbol ≡ indicates that 3 + 4 and 2 are equivalent on the clock. Verify that each of the following is correct:

$$2 + 3 \equiv 0 \pmod 5$$
$$4 + 4 \equiv 3 \pmod 5$$
$$3 + 3 \equiv 1 \pmod 5$$

A second operation may also be introduced on the symbols 0, 1, 2, 3, 4. Since this operation is related in a familiar way to the operation that we have called addition, this new binary operation is called multiplication and the familiar × symbol is used.

By multiplication on this clock we shall mean repeated clockwise rotations. Thus, 2×4 means $4 + 4$ and 4×3 means $3 + 3 + 3 + 3$. Verify that each of the following is correct:

$$2 \times 4 \equiv 3 \pmod 5$$
$$4 \times 3 \equiv 2 \pmod 5$$
$$3 \times 3 \equiv 4 \pmod 5.$$

Here are the completed tables for addition and multiplication in this system. Seventh graders should be encouraged to complete and use--but not memorize--such tables. In a seventh grade class you would normally develop these tables rather than present them in completed form as is done here.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

We can once again explore each of these systems for the various properties discussed earlier. There is one additional property that we need to explore that includes two operations. Consider these problems:

(a) $3 \times (4 + 2) =$
$3 \times 1 =$
3

(b) $(3 \times 4) + (3 \times 2) =$
$2 + 1 =$
3

Remember, we operate within the parentheses first. Note that we obtain the same result either way. Is this a pattern or an accident? Do we always get the same answer if we add before we multiply as we do when we multiply first and then add?

Try each of the following pairs of problems:

(a) $2 \times (3 + 4)$ and $(2 \times 3) + (2 \times 4)$

(b) $4 \times (3 + 2)$ and $(4 \times 3) + (4 \times 2)$

The answers are the same for each pair and will be for all possible similar combinations of three elements. We say that multiplication distributes over addition. In short we call this the distributive property, a property that relates the two operations of addition and multiplication.

> Given a set M and two binary operations * and o defined on M. The operation * distributes over the operation o if
>
> $x * (y \circ z) = (x * y) \circ (x * z)$
>
> for all elements x, y, and z of set M.

Notice that the definition here is quite general and is not limited to a set of numbers and the operations of addition and multiplication. It is true that most of the applications to be encountered by students involve numbers. However, the definition is more general. In fact, when working with sets, the operations of intersection and union are each distributive over the other. You can verify by means of Venn diagrams that intersection distributes over union and also that union distributes over intersection. This is quite different from the case with numbers where multiplication is distributive with respect to addition but addition is not distributive with respect to multiplication.

The distributive property is a very important one for junior high school youngsters to understand. It forms the basis for the work that they do later in algebra in both multiplying and factoring. For example, the distributive property is the justification for such statements as:

$$3a(b + c) = 3ab + 3ac$$

$$x^2y + xy^2 = xy(x + y)$$

$$(x + 4)(x + 3) = x^2 + 7x + 12 .$$

The distributive property also forms the basis for explaining many of the usual arithmetic algorithms. For example, consider the product 9×37:

$$9 \times 37 = 9(30 + 7) = (9 \times 30) + (9 \times 7)$$

In another form:

$$\begin{array}{r l}
37 & \\
\times\ 9 & \\
\hline
63 & \qquad (9 \times 7) \\
\underline{270} & \qquad (9 \times 30) \\
333 &
\end{array}$$

Of course, we abbreviate the work, but it is nevertheless based upon this most important property.

Although this chapter has included several examples that indicate the importance of the various properties developed, most youngsters will have to accept this importance on faith at first. Later on these properties are used more extensively to justify what otherwise would appear as mechanical operations.

An interesting item that can be described to seventh graders is a hypothetical computing machine that has room for only one decimal place. Thus, the machine would compute $.8 \times .7$ as $.56$ and round this off to $.6$; and it would compute $.8 \times .4$ as $.3$. This is an example of a non-associative operation. For example, suppose the machine had to compute $.8 \times .7 \times .6$:

$$(.8 \times .7) \times .6 = .6 \times .6 = .4;$$

whereas $$.8 \times (.7 \times .6) = .8 \times .4 = .3.$$

This helps students see that not all operations obey the various properties listed in this chapter.

Class Exercises

The two tables below describe a mathematical system composed of the set {A,B,C,D} and the two operations * and o.

*	A	B	C	D
A	A	B	C	D
B	B	C	D	A
C	C	D	A	B
D	D	A	B	C

o	A	B	C	D
A	A	A	A	A
B	A	B	C	D
C	A	C	A	C
D	A	D	C	B

27. Do you think * distributes over o ? Try several examples.
28. Do you think o distributes over * ? Try several examples.
29. Find the identity elements for * and o.

4.5 Conclusion

The major objective of the work developed in this chapter has been to achieve some appreciation of the nature of a mathematical system. Each of the properties developed is of importance and will be further explored in the forthcoming chapters of this text. Junior high school youngsters need to see these properties as they relate to familiar sets of numbers as well as to abstract systems. In general, they enjoy and have many opportunities for creativity as they explore these abstract systems.

Summary

A binary operation is a rule whereby to each pair of elements of a set there corresponds exactly one element.

A mathematical system is a set of elements with one or more binary operations defined in the set.

A set is closed under a binary operation if every two elements of the set combined by the operation give a result which is an element of the set.

A binary operation is commutative if, for every two elements, the result of combining them by the operation is independent of the order. If $*$ is the operation and x and y are the elements, then $x * y = y * x$.

A binary operation is associative if, for any three elements, the result of combining the first with the combination of the second and third is the same as the result of combining the combination of the first and second with the third. If $*$ is the operation and x, y, and z are the three elements, then

$$x * (y * z) = (x * y) * z.$$

An identity element for a binary operation defined on a set is an element of the set which does not change any element with which it is combined.

Two elements are inverses of each other under a binary operation if the result of this operation on the two elements is the identity element for that operation.

A binary operation $*$, distributes over the binary operation o provided

$$a * (b \circ c) = (a * b) \circ (a * c)$$

for all elements a, b, c.

Chapter Exercises

1. Complete an addition and a multiplication table for a mod 4 arithmetic using the set of elements $M = \{0,1,2,3\}$. (Compare the entries of your table with those in the tables given for Class Exercises 27-29. Let $A = 0$, $B = 1$, $C = 2$, and $D = 3$.)

2. Use the tables from Exercise 1 to complete each of the following:
   (a) $2 + 3$
   (b) $3 + 3$
   (c) 2×3
   (d) 3×3
   (e) $3 \times (2 + 3)$

3. Answer each of the following about the mathematical system of multiplication (mod 4).
   (a) Is the set closed under the operation?
   (b) Is the operation commutative?
   (c) Do you think that the operation is associative?
   (d) Which elements have inverses, and what are the pairs of inverse elements?
   (e) Is it true that if a product is 0 then at least one of the factors is 0 ?

4. Consider the set of elements {E, O} and the binary operations +, ×, defined as follows:

+	E	O
E	E	O
O	O	E

×	E	O
E	E	E
O	E	O

   (a) Is either operation commutative?
   (b) Is there an identity element for × in this set?
   (c) Is × distributive over + ? That is, does $a \times (b + c) = (a \times b) + (a \times c)$ for all replacements of E or O for a, b, and c ?
   (d) Is + distributive over × ?

For each of the following described sets and operations determine whether the set is closed and find which operations are commutative or associative.

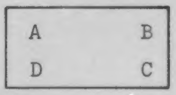
5. Set: All counting numbers less than 50.
   Operation: Multiply the first by 2 and then add the second.
   Example: 3 combined with 5 produces 11 since $2 \cdot 3 + 5 = 11$.

6. Set: All counting numbers.

Operation: Raise the first number to a power whose exponent is the second number.

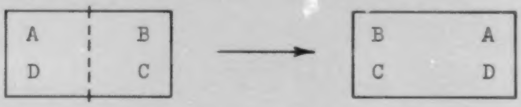
Example: 5 combined with 3 produces 5^3.

7. Consider a system formed as follows. Place an index card, marked as in the diagram, in "standard position":



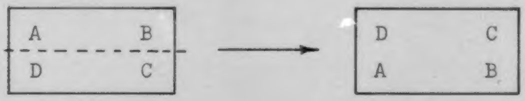
Let us use I to mean leave the card in place. V means to flip the card over using a vertical axis:

V:



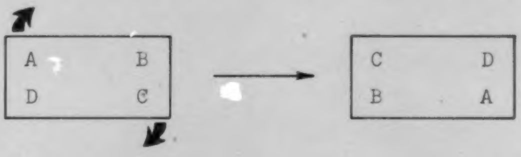
H means to flip the card over using a horizontal axis.

H:



R means to rotate the card halfway around in the direction indicated.

R:



Our set of elements is {I,V,H,R}. The operation will be "ANTH" which means to do one thing "and then" do another. Thus, "H ANTH V" means to flip the card over using a horizontal axis, and then flip the card over again using a vertical axis. Try it with an actual card. You should find that H ANTH V=R.

(a) Complete the following table for the operation ANTH. Some entries are already given for you.

ANTH	I	V	H	R
I	I	V		
V			R	H
H	H	R		
R			V	

Examine the table for the operation ANTH.

(b) Is the set closed under the operation?

(c) Is the operation commutative?

(d) Do you think the operation is associative? Use the operation table to check several examples.

(e) Is there an identity element for the operation ANTH ?

(f) Does each element of the set have an inverse under the operation ANTH ?

8. Let sets A, B, and C be defined as follows:

$A = \{1,2,3,4,5\}$; $B = \{3,4,5,7,8\}$; $C = \{1,3,5,8,9\}$.

(a) Show that the operation $\cup$ (union) distributes over the operation $\cap$ (intersection). That is, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(b) Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Answers to Exercises

1. One less than the sum of the two numbers. $a \circ b = (a + b) - 1$.
Note: This can also be written as $(a - 1) + b$ or $a + (b - 1)$.

2. The average of the two numbers. $a \perp b = \frac{a + b}{2}$.

3. The larger of the two numbers if the numbers are different; that number if they are the same.

4. The sum of twice the first number and the second number. $a \sim b = 2a + b$.

5. Twelve minus the sum of the two numbers. $a \sqcup b = 12 - (a + b)$.

6. a 7. 3 8. b 9. 3 10. Yes 11. Yes

12. Yes. No; $7 \oplus 3 = 5$ whereas $3 \oplus 7 = 3$. You need find only one counter-example to show that commutativity does not hold.

13. No; $c \triangle b = c$ whereas $b \triangle c = a$.

14. Closed, commutative, and associative. (It is understood here that "choose the smaller number" means to select that number if both are the same. That is, 32 combined with 32 produces 32.)

15. Closed and associative; not commutative.

16. b 17. e 18. c 19. Yes 20. Yes 21. Yes; C

22.

Element	a	b	c	d	e
Inverse	e	d	c	b	a

23. 7 24. 10 25. Yes; 12

26. The inverse of 5 is 7 since $5 + 7 = 12$, the identity element for addition.

27. No. For example, $B * (C \circ D) = B * C = D$, whereas $(B * C) \circ (B * D) = D \circ A = A$.

28. Yes. For example, $B \circ (C * D) = B \circ A = A$, and $(B \circ C) * (B \circ D) = C * D = A$.

29. The identity element for $*$ is A.
The identity element for $\circ$ is B.

Chapter 5

INTRODUCING NEW NUMBERS

Introduction

In this chapter we shall examine in detail some of the different number systems that are encountered in the seventh grade. In some respects the treatment will be that as given in a seventh grade course and in other respects the treatment will be a bit more advanced. Though seventh grade mathematics does not normally include a study of negative numbers we shall introduce them in this chapter. There are three reasons for doing so. (a) The introduction of negative numbers is in many respects similar to the introduction of rational numbers and thus strengthens our understanding of this process. (b) Negative numbers are commonly introduced in the eighth grade and junior high school teachers either teach eighth grade or wish to be knowledgeable in the subject matter their students will learn in the following years. (c) Some youngsters will have met the negative numbers in earlier grades, and we may expect to have more such youngsters in the future.

The development of the real number system that we are about to trace in this and the next four chapters is a remarkable achievement of the human mind. These chapters will present the result of over four thousand years of human thought. In our modern age there are many ways in which this development may be carried out. We shall begin with the counting numbers.

5.1 The Counting Numbers and the Whole Numbers

Although the counting numbers are exceedingly abstract, they do not frighten us, for we are very familiar with them. At this particular time let us accept the counting numbers and some of their properties and build on them. The properties of the counting numbers we wish to build on in the beginning are properties of binary operations. The binary operations, addition and multiplication, of the counting numbers were introduced by man to enable him to make greater use of the counting numbers. These binary operations turn out to have some very useful properties. Both addition and multiplication are binary operations which are closed, commutative, and associative. That there are two binary operations which have these three properties is itself useful and interesting, but the utility and interest is much increased by the fact that these operations are inter-related. For the counting numbers we have the distributive property: $a(b + c) = ab + ac$.

What do we mean when we say a property holds? For example, let us look at multiplication. What do we mean when we say multiplication is commutative? Certainly no one has verified all the possible products; it is most unlikely that the product

$$987685948329573 \times 897869697857463957362$$

has ever been computed and equally as unlikely that the product

$$897869697857463957362 \times 987685948329573$$

has been computed. Nevertheless, we assert with complete confidence that the products are the same. We fearlessly make this assertion because we may derive it from our definition of multiplication. In the abstract systems of Chapter 4 we decided that an operation is commutative by examining a table. The table serves as the definition of the operation; it tells you how to operate on two of the elements of the system to produce a resulting element. From the table, which is the definition of the operation, we derive the properties of the operation. Thus, in Chapter 4 a system was shown to be commutative by examining a table.

For the counting numbers there are too many elements to exhibit a multiplication table. To investigate the properties of multiplication we must go back to a definition. Multiplication of counting numbers is best defined in terms of sets, though sometimes it is done as repeated addition. To show that multiplication of counting numbers is commutative we go back to the definition and derive, logically, that the property holds. Such a development of the counting numbers may be found in many sources.

The counting number 1 has a property shared by no other counting number. With respect to multiplication we have

$$1 \cdot a = a \cdot 1 = a$$

where a represents any counting number. In Chapter 4 we learned to call an element with this property an identity element. Since 1 is an identity element with respect to multiplication, it is called a *multiplicative identity*.

Class Exercises

1. We know that 2 is another name for $1 + 1$. Use this fact and a property of the operations on the counting numbers to show that

$$2 \cdot 2 = 2 + 2 .$$

2. The distributive law tells us $3(5 + 6) = 3 \cdot 5 + 3 \cdot 6$. Why is it true that $3(5 + 6 + 8) = 3 \cdot 5 + 3 \cdot 6 + 3 \cdot 8$?

3. What properties of the counting numbers are used to show that $5 \cdot (6 \cdot 9) = 9 \cdot (6 \cdot 5)$?

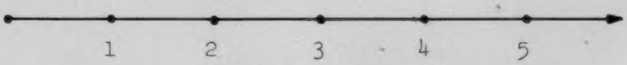
4. Using the properties of the counting numbers, show that $(3 + 4) + (5 + 6) = \big((6 + 4) + 5\big) + 3$.

5. Among the counting numbers an additive identity would be a counting number x with the property that

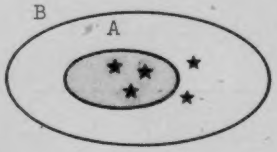
$$x + a = a + x = a$$

for any counting number $\underline{a}$. Do the counting numbers have an additive identity?

The properties we have referred to above are properties of two very special binary operations, addition and multiplication. The counting numbers have other properties that are equally interesting and at least as basic. Among these is the property of order. Given two counting numbers, we say that they are equal or that one is greater than another (or smaller). We may use the number line to represent the order that exists among the counting numbers.



Of course, the idea of order for counting numbers exists independently of their number line representation. For example, if set A is a proper subset of the finite set B ($A \subset B$ where $A \neq B$), then the number of elements in set A is less than the number of elements in set B.



Small children learn this relationship long before they learn to use either numerals or the number line. A child knows that five pieces of candy are better than two long before he knows the meaning of the symbols in the sentence "5 = 2 + 3."

When we adjoin to the set of counting numbers the number 0, we call the collection thus obtained the set of whole numbers. The whole numbers share with the counting numbers many arithmetical properties. For the whole numbers there are two binary operations, addition and multiplication, that are closed, commutative and associative. The distributive law holds, connecting the two operations. The adjunction of the number 0 provides an additive identity to the set. Thus, with respect to addition we have

$$0 + a = a + 0 = a$$

where a represents any counting number.

5.2 Positive Rational Numbers

While the counting numbers have many desirable features, they also suffer from numerous serious deficiencies. There are many elementary questions that we would like to answer, questions that may be asked with counting numbers, but that cannot be answered with counting numbers. Two boys wish to share equally five pieces of candy; how many pieces should each boy receive? Gasoline is 30 cents a gallon; how much gasoline may be purchased for \$2.00 ? A man is able to walk ten miles in four hours; how far can he walk in one hour? These questions and many others are reasonable ones that we wish to answer. However, we have learned to go outside the system of counting numbers to find the answers. Most of us have learned to do this in a piecemeal fashion. We learn about two equal parts of a cake, two equal parts of an apple, and so on, eventually coming to the concept of the number we name $\frac{1}{2}$. The same process leads us to concepts for $\frac{1}{4}$ and $\frac{1}{3}$ and other unit fractions.

In this chapter we wish to give a systematic introduction to rational numbers. To this end we abstract from the problems which lend to rational numbers their common feature. Though rational numbers seem to have evolved in many diverse ways, there is a common feature. Indeed all positive rational numbers are solutions of equations stated in terms of counting numbers. The equations

$$\begin{aligned} 2x &= 5 \\ 30x &= 200 \\ 4x &= 10 \end{aligned}$$

are stated with counting numbers. However, the solutions to the equations given above are not counting numbers. The solutions of these equations provide the answers to the questions of the preceding paragraph. Thus, while the questions seem diverse, the equation approach points out their similarity.

If we restrict ourselves to counting numbers, we would have to say that the solution set of each of the equations $2x = 5$, $30x = 200$, and $4x = 10$ is the empty set. Of course, we may write equations that have nonempty solution sets in terms of counting numbers.

$$3x = 6$$
$$9x = 54$$
$$19x = 513$$

Our state of mind at this point may be compared to a carpenter who has a rule marked in inches without further subdivisions. He is able to work as long as the measurements he needs are full inches. Since most lumber does not measure a whole number of inches he is soon apt to run into trouble. Our mythical carpenter may well do what many small children and some adults do; invent markings. To transfer measurements, a child will ignore the markings between the inch markings on his ruler and make a pencil mark. Real carpenters with real rules will also invent markings. Listen carefully to good carpenters talking and you will hear such things as "a fat seven and three sixteenths." Their rules are not adequate for their needs, and they invent new quantities, perhaps not very precise, but sufficient for their needs.

There are many ways and levels through which rational numbers may be introduced; through experience, as children learn them, or through ordered pairs, as some texts and most mathematicians do them. We will take a middle course that ties in with seventh grade SMSG texts.

We will "invent" numbers to serve us better in certain real life situations. The concepts of $\frac{1}{2}$, $\frac{3}{4}$, $\frac{8}{12}$, and so on, are introduced physically using candy bars, cakes, pies, glasses of milk, and the like. A candy bar split into two equal pieces conveys the idea of $\frac{1}{2}$ to a child. To us the equation $2x = 1$ will carry the same concept of $\frac{1}{2}$.

A pie is cut into four equal pieces, one of which is then eaten. How much remains? Again to us, the concept is probably clear through the equation $4x = 3$. However, the child needs the physical examples to strengthen the concept of $\frac{3}{4}$. As the child develops, we point out to him that $\frac{3}{4}$ is the name of a number with the property that $4 \cdot \frac{3}{4} = 3$. In other words, it is the solution to the equation $4x = 3$.

Likewise, $\frac{8}{12}$ is a name for the number with the property that $12 \cdot \frac{8}{12} = 8$. It is the solution to the equation $12x = 8$.

Actually, we invent mentally a class of numbers that are solutions of equations in the form

$$bx = a \quad (a, b \text{ counting numbers}).$$

Once this is assimilated, fractions, rational numbers, and their relationship to each other lose their mystery.

We have mentally invented numbers that are solutions of equations of this special form. To continue our discussion it will certainly be convenient to have names for these numbers. Collectively we call them the positive rational numbers. The individual numbers are solutions of equations and can be named from the equations. The number that is a solution of $2x = 1$ is named $\frac{1}{2}$. The number that is the solution of $34x = 38$ is named $\frac{38}{34}$, and in general, the number that is a solution of $bx = a$, a and b counting numbers, is named $\frac{a}{b}$. In each case we introduce a symbolic name, a fraction, for the concept of a rational number.

Class Exercises

6. For each equation, give the solution in the form $\frac{a}{b}$, with a and b whole numbers.

(a) $3x = 11$	(c) $10x = 93$
(b) $6x = 15$	(d) $8x = 9$

7. Write an equation for which each of the following is the solution.

(a) $\frac{2}{7}$	(c) $\frac{5}{12}$
(b) $\frac{3}{4}$	(d) $\frac{90}{100}$

Let us pause to reflect on this introduction to positive rational numbers. The child learns about rational numbers through physical experiences. Most texts build upon these experiences to show that a rational number is the solution of an equation. Thus, a child will agree that $\frac{3}{4}$ has the property that $4 \cdot \frac{3}{4} = 3$. We have adopted a different view. We started with the equation and introduced the solution. The end result of the two methods will be the same.

We have invented some numbers and given them names. So far they have only one attribute; they may be used with counting numbers to make true sentences. We wish to take the positive rational numbers and use them to form an algebraic system. This will be done in Chapter 6 where two binary operations will be introduced on the set of positive rational numbers. Though we wish to give an abstract development of rational numbers, we shall not ignore our previous, less formal, knowledge. Rather we shall use this knowledge to suggest the direction of our abstract approach.

5.3 Equivalent Fractions

Some of the problems that concern us here should definitely not be made the concern of junior high school students. Statements that those students accept without hesitation will be examined in detail here. The development is given for the teacher, to help shed light on the structure of the rational number system.

From previous experience with rational numbers, everyone accepts the statement that $\frac{1}{2}$ and $\frac{2}{4}$ name the same rational number. Let us see how this conclusion may be reached without cutting a cake into four parts. From our notation, $\frac{1}{2}$ names a solution of the equation

$$2x = 1$$

while $\frac{2}{4}$ names a solution of the equation

$$4y = 2.$$

To avoid prejudging the matter, we have used x in one equation and y in the other, thus in no way implying the two solutions are necessarily the same. The equation $2x = 1$ has a solution which we have named $\frac{1}{2}$; that is, $2 \cdot \frac{1}{2}$ and 1 are two names for the same number and we write

$$2 \cdot \frac{1}{2} = 1.$$

Since $2 \cdot \frac{1}{2}$ and 1 name the same number, the products $2 \cdot (2 \cdot \frac{1}{2})$ and $2 \cdot 1$ will also name the same number:

$$2 \cdot (2 \cdot \frac{1}{2}) = 2 \cdot 1 = 2.$$

Students will readily accept that this implies:

$$(2 \cdot 2) \cdot \frac{1}{2} = 2$$

$$4 \cdot \frac{1}{2} = 2.$$

But the last equation above is in the form of $4y = 2$.

121

Thus, it is concluded that the solution of

$$2x = 1$$

is also the solution of

$$4y = 2\ ;$$

or that $\frac{1}{2}$ and $\frac{2}{4}$ name the same rational number.

If we carefully look at this reasoning, we see a serious gap. We have fallen into the error of stating

$$(2 \cdot 2) \cdot \frac{1}{2} = 2 \cdot (2 \cdot \frac{1}{2}).$$

We have inadvertently used the associative law in a situation where there is no justification for its validity.

Let us pause once again for some reflection. We are attempting to adopt a state of mind in which we invent for ourselves the positive rational numbers and some properties of these new numbers. To do this we must be careful that we do not use a property that is not of our making. We do not regard ourselves as completely free in our invention for experience has taught us that certain properties of binary operations are most useful. Thus, we shall aim for a system with two binary operations that are associative and commutative. Furthermore, we shall want these operations, if possible, to be connected by a distributive law.

Thus, we want to construct a system in which $2 \cdot (2 \cdot \frac{1}{2}) = (2 \cdot 2) \cdot \frac{1}{2}$. There is no guarantee that such can be done. However, from our previous arguement we do know it will be impossible to have an associative binary operation unless we agree that $\frac{1}{2}$ and $\frac{2}{4}$ name the same number.

While the teacher should be aware of the developments of the preceding paragraphs, their content is not appropriate for junior high school students.

By the same reasoning we are led to the conclusion that $\frac{2}{3}$, $\frac{8}{12}$, $\frac{18}{27}$, and $\frac{32}{48}$ should name the same number. That is, the solutions of each of the following equations are equal:

$$\begin{array}{ll} 3x = 2 & 27z = 18 \\ 12y = 8 & 48x = 32. \end{array}$$

Before stating a formal definition of equality let us look at another example. The numbers $\frac{4}{36}$ and $\frac{3}{27}$ are solutions of the equations $36x = 4$ and $27y = 3$, respectively; $36 \cdot \frac{4}{36} = 4$ and $27 \cdot \frac{3}{27} = 3$. To compare these two statements, let us multiply the first equation by 27 and the second equation by 36:

$$27 \cdot (36 \cdot \frac{4}{36}) = 27 \cdot 4$$

$$36 \cdot (27 \cdot \frac{3}{27}) = 36 \cdot 3.$$

The numbers 27 and 36 were chosen to give uniformity to the left-hand sides of the two statements. Both now contain the product of the two factors 27 and 36 times the respective numbers $\frac{4}{36}$ and $\frac{3}{27}$. The right-hand sides, $27 \cdot 4$ and $36 \cdot 6$, both name 108. Thus, we see that $\frac{4}{36}$ and $\frac{3}{27}$ must both be solutions of the same equation, $(27 \cdot 36) \cdot x = 108$, and hence should be called names for the same number. How may we decide when two symbols name the same rational number?

Definition: The symbols $\frac{a}{b}$ and $\frac{c}{d}$ name the same rational number if it is true that:

$$bdx = ad$$

and

$$bdx = bc$$

are the same equation.

This definition is applied in several examples that follow:

Example: Do the symbols $\frac{18}{64}$ and $\frac{27}{96}$ name the same rational number? By direct application of the definition we get

$$(64 \cdot 96)x = 18 \cdot 96 \quad \text{or} \quad 6144x = 1728$$

and

$$(64 \cdot 96)x = 64 \cdot 27 \quad \text{or} \quad 6144x = 1728.$$

Since the equations are identical, we conclude the two named rational numbers are equal.

Example: Do $\frac{112}{64}$ and $\frac{602}{300}$ name the same rational number? We examine the equations

$$(64 \cdot 300)x = 112 \cdot 300 \quad \text{or} \quad 19{,}200x = 33{,}600$$

and

$$(64 \cdot 300)x = 64 \cdot 602 \quad \text{or} \quad 19{,}200x = 38{,}528.$$

As these equations are not identical, we conclude the two named rational numbers are not equal.

Notice that in each case the actual test of equality is made by comparing equations. The teacher should understand that this development is given here to emphasize the importance of the equation approach to defining rational numbers. On the other hand, it is apparent from the above example that the test of equality for the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ can be made simply by comparing the products ad and bc for equality since these are the products that appear on the right side of the equations. Hence, it is more common for the junior high school student to compare rational numbers using the following definition:

> The symbols $\frac{a}{b}$ and $\frac{c}{d}$ name the same rational number if and only if $ad = bc$.

The teacher should see clearly the comparison between this definition and the previous one as it will help to give background and understanding to the teaching of rational numbers and proportions in the junior high school.

Symbols such as $\frac{2}{3}$, $\frac{9}{12}$, and $\frac{11}{17}$ name rational numbers since each is the solution to an equation in the form $bx = a$ where a and b are counting numbers. Symbols in the form $\frac{a}{b}$ that represent the indicated quotient of two quantities are called fractions. When convenient we use the well-known terminology "numerator, denominator" with fractions. If in the fraction $\frac{a}{b}$ both a and b are counting numbers, then the fraction names a positive rational number.

We name rational numbers with fractions; each rational number has many fractional names. Fractions that name the same rational number are called equivalent fractions. Now that the language of fractions and rational numbers has been developed we will not hesitate to say: The number $\frac{1}{2}$ in place of the number which is named by $\frac{1}{2}$.

From Class Exercise 11 we see that every counting number is a positive rational number. (The set of counting numbers is a subset of the set of positive rational numbers.) Counting numbers have many names; some of the names are fractions.

Class Exercises

Note: Exercises 9 through 12 are an essential part of the development of this section.

8. (a) Do $\frac{3}{4}$ and $\frac{9}{12}$ name the same rational number?

   (b) Do $\frac{31}{64}$ and $\frac{85}{175}$ name the same rational number?

9. We know $2 \cdot 100 = 5 \cdot 40$. May we conclude from this statement that $\frac{2}{5}$ and $\frac{40}{100}$ name the same number?

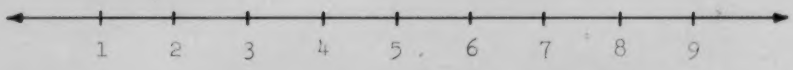
10. If $ad = bc$ (a, b, c, d counting numbers), may we conclude that $\frac{a}{b}$ and $\frac{c}{d}$ name the same number?

11. Do the symbols 4 and $\frac{4}{1}$ name the same number?

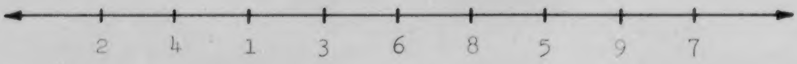
12. Use the definition to show that $\frac{1}{3}$ and $\frac{11}{33}$ name the same rational number. The equation for $\frac{1}{3}$ may be written as $11 \cdot 3x = 11 \cdot 1$. Does this equation show that $\frac{1}{3}$ and $\frac{11}{33}$ represent the same number?

5.4 Order

A valuable representation of the counting numbers is the number line. Can every positive rational number be represented as a point of a number line? The counting numbers are usually represented as follows:



There are other ways to represent the counting numbers on a line as in this figure:



It is clear that every counting number may be represented as a point of the line in this pattern. Nevertheless, we normally reject this representation. It is rejected as it does not contain the information on order indicated by the normal representation. The concept of order also exists for the rational numbers and we would like to indicate this with a number line representation of rational numbers. Order is another aspect of rational numbers of which we have intuitive ideas. Let us make the intuitive ideas, gained from experience, precise.

The numbers $\frac{1}{2}$ and $\frac{3}{4}$ are solutions of the equations $2x = 1$ and $4x = 3$, respectively. We see that $\frac{1}{2}$ and $\frac{3}{4}$ are not equal (meaning they are not names for the same number) since the equations

$$(2 \cdot 4)x = 1 \cdot 4 \quad \text{or} \quad 8x = 4 \quad (x = \tfrac{1}{2})$$

and

$$(2 \cdot 4)x = 2 \cdot 3 \quad \text{or} \quad 8x = 6 \quad (x = \tfrac{3}{4})$$

are not identical.

Since $\frac{1}{2}$ is a solution of $2x = 1$ and of $8x = 4$, while $\frac{3}{4}$ is a solution of $4x = 3$ and of $8x = 6$, we have $8 \cdot \frac{1}{2} = 4$ and $8 \cdot \frac{3}{4} = 6$. Reasoning informally, we might say eight times three-fourths is more than eight times one-half, and hence that three-fourths is greater than one-half.

Let us look at another example. The numbers $\frac{13}{23}$ and $\frac{9}{17}$ are solutions of the equations $23x = 13$ and $17x = 9$, respectively. Do the fractions $\frac{13}{23}$ and $\frac{9}{17}$ name the same number? We examine the equations

$$(23 \cdot 17)x = 13 \cdot 17 \quad \text{or} \quad 391x = 221$$

and

$$(23 \cdot 17)x = 23 \cdot 9 \quad \text{or} \quad 391x = 207.$$

The equations are not equal since 221 does not equal 207. Therefore, we conclude that $\frac{13}{23} \neq \frac{9}{17}$. Further, since 391 times $\frac{13}{23}$ is greater than 391 times $\frac{9}{17}$, we reason intuitively that $\frac{13}{23}$ is greater than $\frac{9}{17}$.

We see from these last two examples that the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ can be ordered by comparing the products ad and bc.

This reasoning guides us to a formal definition.

Definition: Let $\frac{a}{b}$ and $\frac{c}{d}$ be rational numbers with a, b, c, and d counting numbers. If $ad > bc$, we say $\frac{a}{b} > \frac{c}{d}$, read "$\frac{a}{b}$ is greater than $\frac{c}{d}$."

The reader may complain that the definition is difficult to remember. Some practice with it will help and one may always return to the test for equality and reason as we did above.

Class Exercises

Note: Exercises 15 and 16 are an essential part of the development of this section.

13. Insert in the box the proper sign $<$, $>$, or $=$ to make true statements.

(a) $\frac{3}{4} \square \frac{9}{16}$ (c) $\frac{101}{12} \square \frac{67}{8}$

(b) $\frac{27}{81} \square \frac{18}{55}$ (d) $\frac{10001}{59} \square \frac{63}{2}$

14. Insert in the box the proper sign $<$, $>$, or $=$ to make true statements.

(a) $\frac{18}{25} \square \frac{9}{11}$ (d) $\frac{18}{25} \square \frac{108}{132}$

(b) $\frac{18}{25} \square \frac{27}{33}$ (e) $\frac{9}{11} \square \frac{27}{33}$

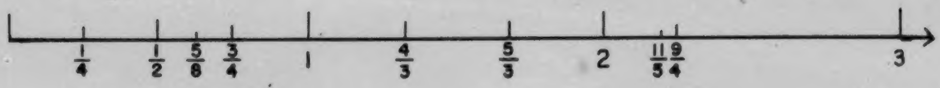
(c) $\frac{18}{25} \square \frac{90}{110}$ (f) $\frac{9}{11} \square \frac{90}{110}$

(g) $\frac{9}{11} \square \frac{108}{132}$

15. Let a, b, and k be counting numbers. Show that $\frac{a}{b} = \frac{ak}{bk}$.

16. Let $\frac{e}{f}$ and $\frac{c}{d}$ be fractions such that $\frac{c}{d} < \frac{e}{f}$. Let k be a counting number and show that $\frac{ck}{dk} < \frac{e}{f}$.

Exercises 15 and 16 partially show that the order relation of the rational numbers does not depend on the particular fractional representation. Now that a definition of order has been introduced we may systematically make rational numbers correspond to points on a number line. As an example let us search for a point corresponding to $\frac{9}{4}$. It is readily shown that $2 = \frac{2}{1} < \frac{9}{4}$ while $\frac{9}{4} < \frac{3}{1} = 3$. That is, we would like the point representing $\frac{9}{4}$ to be between the points whose coordinates are 2 and 3. Which point between 2 and 3 should we choose? Here we fall back upon our idea of measure on a line. (See Chapter 12.)

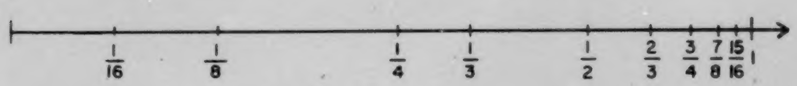


Here the visual representation of the positive rational numbers on the number line indicates the relative magnitude of the rational numbers. Each $\frac{1}{4}$ unit corresponds to the same distance.

A more customary approach to the ordering of the rational numbers and to the number line begins with the number line itself. For example, the line between 0 and 1 is divided into four parts of equal length. The end points of the parts are labelled $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1. Also, the same segment is divided into five parts of equal length with end points labelled $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, and 1. By inspection, we determine that $\frac{4}{5}$ is greater than $\frac{3}{4}$. This procedure becomes unmanageable for such fractions as $\frac{131}{725}$ and $\frac{654}{4312}$. Thus, we are forced to use the more sophisticated approach of our definition. For children, the geometric or physical introduction is recommended, quickly followed by the algebraic approach.

Class Exercises

17. Here is a possible correspondence of rational numbers and points on a line. Criticize this correspondence.



5.5 Whole Numbers and Rational Numbers

We have confined ourselves to equations of the form $bx = a$, (a, b counting numbers). Let us extend our horizons and examine equations of the form $bx = a$ where we now let a and b be whole numbers. Though we now have changed the setting of our discussion to include zero, much remains familiar. All the equations involving counting numbers are still with us. There are, however, some new equations. Some examples of these are:

$2x = 0 \qquad 0w = 0$

$0y = 17 \qquad 0x = 5$

$3z = 0 \qquad 5x = 0$

Let us look at some of these examples, recalling that the product of 0 and any number is 0. This latter fact makes us rule out equations of the form $0y = 17$; $0x = 5$; $0x = a$ where $a \neq 0$. The reader should become indignant at the suggestion that we rule out the equation $0x = 5$ and should demand that more new numbers be invented so as to solve equations of this form.

Indeed, it would be possible to make up such new elements; however, what are the consequences? We would, of course, have to give up the result that the product of 0 and any number is 0. We would have to give up the distributive law which could no longer hold. The whole structure of arithmetic would collapse. The gain is not worth what would be lost and so we do not allow such invention. Since we now exclude the possibility of solutions to $0x = a$ where $a \neq 0$, we are in essence saying that the corresponding symbol $\frac{a}{0}$ has no meaning.

Equations of the form $0x = 0$ do have solutions. Since the product of 0 and any number is 0 we may substitute any number for x to obtain a true statement. But the equation $0x = 0$ does not define any unique number and so these equations are also ruled out. Thus, the corresponding symbol $\frac{0}{0}$ also has no meaning. Indeed, these last two results lead us to the statement: We cannot divide by zero.

Finally, there are the equations of the form $2x = 0$, $3z = 0$, $5x = 0$, $bx = 0$, (b a counting number). These equations have 0 as a solution. With our original notation for fractions, we denote the solution of $2x = 0$ with $\frac{0}{2}$ and the solution of $ax = 0$, $a \neq 0$, with $\frac{0}{a}$.

Class Exercises

18. Use the definition to show that the fractions $\frac{0}{2}$ and $\frac{0}{5}$ name the same rational number and that this number must be identified with 0.

19. If the product of two numbers is 0 in multiplication mod 5, must one of the factors be zero? Answer the same question for multiplication mod 4.

5.6 The Integers

In Section 2 of this chapter we observed that the counting numbers do not provide a system rich enough to contain solutions to equations such as $3x = 4$. This provided the opportunity to introduce some new numbers, the positive rational numbers. If we return again to the counting numbers, we find another class of questions stated in terms of the counting numbers that cannot be answered with counting numbers (or with positive rational numbers). The questions or equations which were used to introduce the rational numbers were multiplicative in nature; now we look at those which are additive.

Here are some questions:

(a) John is now 12 years old. How old will he be 16 years from now?

(b) Mary had 5 Beatle records. She received 4 more for her birthday. How many Beatle records does she now have?

(c) Mrs. Smith has 2 books of trading stamps. She wishes to obtain a three-piece towel set which requires 97 books of stamps. How many more books does Mrs. Smith need?

(d) The constitution requires that the President of the United States be 35 years old. John is now 27. In how many years will he be eligible to be president?

The answers to (a) and (b) are obtained by using the binary operation of addition on the counting numbers. Problems (c) and (d) may also be phrased as addition problems:

What number when added to 2 yields the sum 97 ?

What number when added to 27 yields the sum 35 ?

Put in terms of open sentences, we want the solution sets of the equations (open sentences)

$$2 + x = 97 \quad \text{and} \quad 27 + x = 35.$$

These are the equations we want solved. However, we have learned a systematic attack on such problems through subtraction and we immediately fall back upon it by solving

$$97 - 2 = x \quad \text{and} \quad 35 - 27 = x.$$

There are other questions that may be asked in the framework of counting numbers.

(e) What is the solution set of $9 + x = 4$?

(f) What is the solution set of $5 + x = 5$?

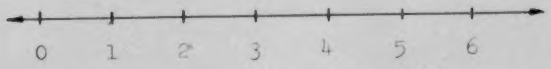
If we confine ourselves to the counting numbers, we would have to answer that the solution sets of (e) and (f) are the empty set. That is, there is no counting number that may be added to 9 to produce the sum 4, nor is there any counting number that may be added to 5 to produce the sum 5. (Recall that 0 is not a counting number.)

This presents a most unsatisfactory situation; some equations like $4 + x = 9$ have nonempty solution sets while others like $9 + x = 4$ have empty solution sets within the framework of the counting numbers. Yet we frequently want solutions to problems that take the form of question (e). For example:

(e') Mary has 9 Beatle records. Her father can tolerate only 4 of them. What can the father do to make the situation tolerable? A drastic solution would have the father destroy 5 records.

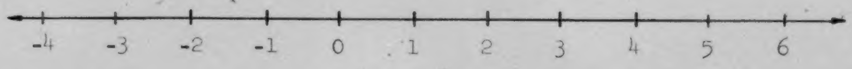
(e") Marvin asked for directions. He was told to make a right turn at the fourth light. Through an oversight Marvin went to the ninth light. Is there any way for Marvin to return to the fourth light? (U-turns are allowable.)

Again we wish to develop numbers to answer the questions we can ask. However, there is a difference this time. The student generally has had some experience with the solution of the equation $2x = 1$, while rarely does the student have any experience with a solution of the equation $x + 2 = 1$. Thus, before a study of equations can be successfully started, some informal background experience is useful. This is commonly done by returning to the number line for counting numbers. Recall that certain uniformly spaced points on the line corresponding to the counting numbers are singled out and named 1, 2, 3, ... One other point has been marked and named 0.



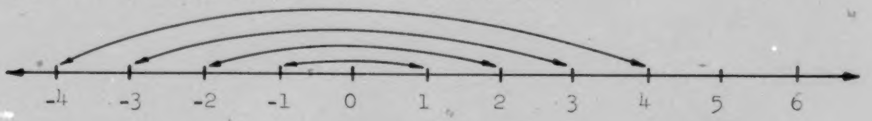
We will now extend the number line to the left. Many devices are used to justify naming points to the left of 0; thermometers, bank accounts, altitude above and below sea level, and distance. Let us simply say that points on one portion of the line have been named and that we wish to name points on the other portion. We could use I, II, III, IV, V, VI, ...

It is more convenient and much more useful to make use of the Hindu-Arabic numerals. In order to be able to differentiate between those naming points to the right of 0 and those to the left of 0 we use the symbol "$^{-}$" to denote numerals corresponding to points on the left.

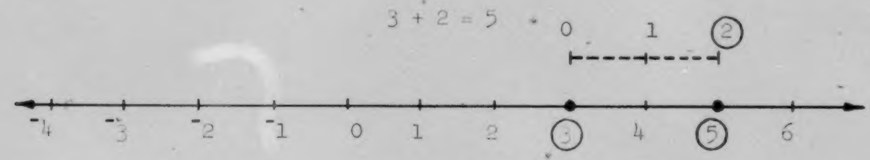


Frequently points to the right of 0 on the number line are named with the symbol "+" to emphasize the distinction between these and the ones on the left of 0. For example, $^{-}1$ names a point one unit to the left of 0 while +1 names a point one unit to the right. These symbols are read "negative one" and "positive one", respectively. Of course, 1 and +1 are just two different ways of naming the number 1 while 1 and $^{-}1$ name two different numbers.

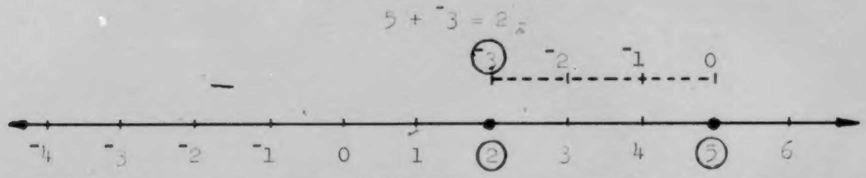
Note that on the number line we have now located a point opposite to each counting number: $^{-}1$ is the opposite of 1, $^{-}2$ is the opposite of 2, and so on. For every counting number b, there is a corresponding negative number, ^{-}b. The opposite of 0 is 0 itself.



Recall that the number line is admirably suited for describing addition of counting numbers. To add counting numbers we "add" corresponding segments.



Now if we interpret "$^{-}$" as meaning we go that amount to the left, we may perform "additions" of other segments (all segments begin at 0).



The figure above indicates the addition using the segment between 0 and 5 and the segment between 0 and $^{-}3$.

Class Exercises

20. Use line segments to perform the following additions:

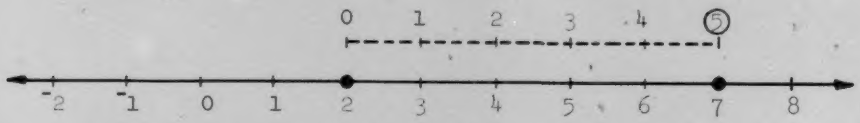
(a) $3 + 4 =$ ______ (d) $^{-}6 + 5 =$ ______

(b) $4 + {}^{-}1 =$ ______ (e) $^{-}2 + {}^{-}4 =$ ______

(c) $5 + {}^{-}6 =$ ______ (f) $^{-}3 + 3 =$ ______

With this interpretation of combining segments we now have a physical method to solve equations of the form $a + x = b$, a and b counting numbers. This is comparable to the aids used to learn about fractions.

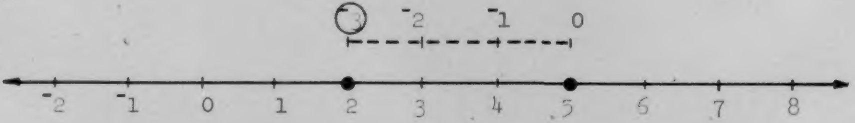
Example: Solve the equation $2 + x = 7$.



$$2 + 5 = 7$$
$$x = 5.$$

In other words, we must move 5 units to the right from 2 in order to reach 7.

Example: Solve the equation $5 + x = 2$.



$$5 + {}^{-}3 = 2$$
$$x = {}^{-}3$$

In this case, we must move 3 units to the left from 5 in order to reach 2.

Class Exercises

21. Use the method of the above examples to solve the equations:

(a) $2 + x = 11$

(b) $5 + x = 6$

(c) $2 + x = 1$

(d) $5 + x = 5$

(e) $8 + x = 8$

(f) $8 + x = 2$

We see from the class exercises that solutions to equations of the form $e + x = f$, e and f counting numbers, can be positive, zero, and negative. The collection of all solutions to equations of this form is called the set of integers. Each member of the set is called an integer.

The set of integers is sometimes represented in the following form:

$$I = \{\ldots, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3, \ldots\}.$$

We see that the set of integers consists of:

the set of counting numbers, $\{1,2,3,\ldots\}$,
called the positive integers;

the number zero, 0 ;

the opposites of the set of counting numbers,
$\{{}^{-}1,{}^{-}2,{}^{-}3,\ldots\}$, called the negative integers.

The subset of the integers which consists of the counting numbers and the integer 0 is called the whole numbers.

The extended number line naturally introduces an ordering of the integers. Given two integers we locate them on the extended number line and call the one to the right the greater. Thus, we call $2 > {}^{-}4$ and ${}^{-}17 > {}^{-}93$. The latter example sometimes causes uneasiness among students. The essential point is that order on the number line involves direction, which is the extension of the notion of order of the counting numbers.

Insert in the box the proper sign $=$, $>$, or $<$ to make true statements:

(a) ${}^{-}4 \square 4$

(b) ${}^{-}3 \square {}^{-}2$

(c) ${}^{-}3 \square {}^{-}4$

(d) ${}^{-}19 \square {}^{-}17$

(e) $6 + {}^{-}7 \square 7 + {}^{-}6$

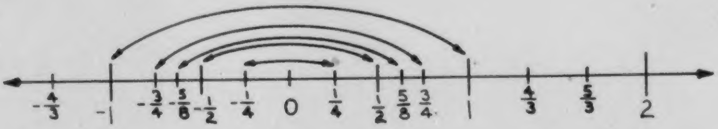
(f) $4 + {}^{-}1 \square 3$

(g) $5 + {}^{-}7 \square 2$

(h) $7 + {}^{-}11 \square 9 + {}^{-}13$

Our discussion of the rational numbers in the previous sections developed only the non-negative rationals. With our knowledge of the integers we now can complete the set of rational numbers by including negative rationals. Recall that the set of whole numbers was extended, by including their opposites, to form the set of integers. In like fashion we will extend the set of positive rational numbers by including their opposites.

Hence, to each positive rational number there is a corresponding negative rational number. Some of these opposites are shown on the number line below.



The complete set of rational numbers now includes all positive rational numbers, all negative rational numbers, and zero.

5.7 Ordered Pairs

It should be clear from our discussion in this chapter that each positive rational number can be introduced simply as an ordered pair of counting numbers; that is, by a pair of counting numbers with the elements of the pair distinguished as to first and last numbers. The notation (a,b), a and b counting numbers, is used to denote an ordered pair. Thus, we can represent a rational number like two-thirds by an ordered pair $(2,3)$ as well as by a fraction $\frac{2}{3}$. Likewise, nine-halves can be represented as $(9,2)$ as well as $\frac{9}{2}$. With this ordered pair notation, it is clear which member of the pair is the first member and which is the last. The ordered pair $(2,3)$ names a different number from the ordered pair $(3,2)$ just as the fractions $\frac{2}{3}$ and $\frac{3}{2}$ name different numbers.

We say that two pairs, (a,b) and (c,d), are equivalent if $ad = bc$. A rational number is a set of all equivalent ordered pairs. The ordered pairs

$$(3,4), \quad (9,12), \quad \text{and} \quad (75,100)$$

all represent the same rational number as do the corresponding fractions

$$\frac{3}{4}, \quad \frac{9}{12}, \quad \text{and} \quad \frac{75}{100}.$$

The method of Section 2 that develops the rationals by the equation method is essentially that of SMSG while the method of ordered pairs of this section is suggested in some other elementary texts. We believe the equation method to be the most satisfactory for young students, but for completeness include the ordered pair method.

In the equation method the experience of the student is used to motivate a belief in the solution of certain equations. An ordered pair approach also uses the student's experience, but in a more formal way.

Class Exercises

23. Write each fraction using the ordered pair notation.

(a) $\frac{3}{8}$ (c) $\frac{5}{3}$

(b) $\frac{9}{16}$ (d) $\frac{1}{100}$

24. Indicate which ordered pairs name the same rational number.

(a) (3,5) and (6,10) (c) (6,6) and (9,9)

(b) (5,7) and (7,5) (d) (5,4) and (15,12)

We have seen how the positive rational numbers can be defined in terms of solutions to equations of the form $bx = a$ while the integers can be defined in terms of solutions to equations of the form $b + x = a$, where in both cases a and b are counting numbers.

To make the analogy between the introduction of positive rational numbers and the integers complete, we may do the following. Let us say that we mentally construct a solution to the equation $7 + x = 3$. We know that among the counting numbers there is no solution. However, we have a physical interpretation of a solution on the number line. Suppose we denote the solution of $7 + x = 3$ by $3 \# 7$ (say "three sharp seven"). In so doing, we are saying that $3 \# 7$ has the property that

$$7 + (3 \# 7) = 3.$$

Just as $\frac{3}{7}$ may be interpreted as representing 3 of 7 equal parts of a circle, we may think of $3 \# 7$ as a name of the point obtained by performing the addition $3 + {}^{-}7$ on the number line.

Are there other equations that have the same solution as $7 + x = 3$? Consider the equation $34 + x = 30$. We can denote its solution by $30 \# 34$ since

$$34 + (30 \# 34) = 30.$$

On the number line the solution may be obtained by performing the addition $30 + {}^-34$. However, we find on the number line that the point named from the addition $30 + {}^-34$ is the same as that named from the addition $3 + {}^-7$. Thus, our invented names for these solutions, $3 \# 7$ and $30 \# 34$, must represent the same number, ${}^-4$. In other words,

$$3 + {}^-7 = 30 + {}^-34 = {}^-4.$$

The solutions to $7 + x = 3$ and $34 + x = 30$ are equal. Here we have the same situation as for the positive rational numbers. There are many equations that lead to numbers we would like to call the same. This is handled in precisely the same way. We agree that different symbols may be different names for the same number. For fractions that name solutions of multiplicative equations the determination is made in terms of products of counting numbers. Two fractions $\frac{a}{b}$, $\frac{c}{d}$, name the same number if and only if $ad = bc$. For additive equations, equations of the form $a + x = b$, a and b counting numbers, a determination is made in terms of sums of counting numbers. If $(a + {}^-b)$ and $(c + {}^-d)$ are names of solutions of $b + x = a$ and $d + y = c$, respectively, (a, b, c, d counting numbers), then they represent the same number if and only if

$$a + d = c + b.$$

While the rotation $3 \# 7$, $30 \# 34$, and $7 \# 3$ is, of course, not standard mathematical notation, it does help to emphasize to the reader that the integers can also be treated as ordered pairs of whole numbers. The ordered pairs used in $3 \# 7$ and $30 \# 34$ represent the same integer while the ordered pairs in $3 \# 7$ and $7 \# 3$ do not.

Class Exercises

25. Indicate which of the two sums represent the same integer.
    (a) $3 + {}^-4$ and $54 + {}^-51$
    (b) $17 + {}^-28$ and $84 + {}^-96$
    (c) $12 + {}^-12$ and $17 + {}^-17$
26. Indicate if the two symbols name the same integer.
    (a) $(11 + {}^-7)$ and 4 (b) $(81 + {}^-99)$ and ${}^-18$ (c) $(10 + {}^-12)$ and ${}^-2$

5.8 Historical Note

In introducing the positive rational numbers before the integers we have followed historical precedence. Sometime before 1700 B.C. the Egyptians were using positive rational numbers. We have been able to date this knowledge due to the discovery of several Egyptian manuscripts. The best known of these is called the Rhind papyrus. An excellent outside assignment would be a report on the Rhind papyrus. (The Encyclopedia Britannica, 11th edition, is a fine source.) Though less well known, the Babylonians of 4,000 years ago also had a knowledge of rational numbers.

The development of the integers came much later, as far as we know. When discussing the origin of ideas one must remember that civilization is not static. Many great nations with complex societies have come and gone. Relics of these people are hard to find, if indeed any relics still exist. Knowledge and libraries are always the targets of despots. The library at Alexandria was wantonly destroyed. It is said that Shih Huang Ti, the emperor of China in 221 B.C. ordered all books of learning destroyed. You will be able to supply some modern instances of attempts to destroy knowledge. Nature also conspires against the preservation of knowledge. Manuscripts written on bark do not long survive.

There is evidence that an appreciation of the integers was developing in the fifth century A.D. Another 1000 years were to pass before the integers were completely absorbed. A complete and rigorous development of our number system was not given until the 18th century.

A modern development of the number system would not follow the historical pattern of development. Rather, one would, after introducing the counting numbers, proceed to the integers. From the integers one would develop the rational numbers (positive and negative) and then go on to the system known technically as the real numbers.

Chapter Exercises

1. Show that $4(5 + 6 + 7 + 8) = (4 \cdot 5) + (4 \cdot 6) + (4 \cdot 7) + (4 \cdot 8)$.

2. Show that $(((3 + 4) + 9) + 5) + 2 = 3 + (4 + (9 + (5 + 2)))$.

3. Give two other symbols which name the same rational number as does $\frac{51}{64}$.

4. What is improper about an improper fraction?

5. Does $\frac{11}{5}$ name a solution of $35x = 77$? Does $\frac{7}{9}$ name a solution of $21x = 27$?

6. Show that $\frac{3}{4}$ and $\frac{15}{20}$ name the same rational number. Show that $\frac{3}{4}$ and $\frac{27}{36}$ name the same rational number.

7. What counting number may be used to name $\frac{6}{1}$? $\frac{9}{1}$? $\frac{103}{1}$?

8. Is there a counting number that may be used to name $\frac{6}{2}$? $\frac{8}{4}$? $\frac{93}{3}$? $\frac{ak}{k}$?

9. Order the following rational numbers beginning with the smallest:

   $\frac{3}{2}, \frac{3}{17}, \frac{3}{1}, \frac{3}{6}, \frac{3}{8}, \frac{3}{5}$.

10. Order the following rational numbers beginning with the smallest:

    $\frac{2}{7}, \frac{9}{7}, \frac{6}{7}, \frac{7}{7}, \frac{18}{7}, \frac{5}{7}, \frac{14}{7}$.

11. Order the following rational numbers beginning with the smallest:

    $\frac{8}{9}, \frac{11}{12}, \frac{19}{20}, \frac{14}{15}, \frac{99}{100}$.

12. Which of the following statements are true?

    (a) The integers are opposites of the counting numbers.

    (b) Zero is an integer.

    (c) The set of whole numbers includes only the positive integers.

    (d) The integer $^-17$ is less than the integer $^-15$.

    (e) Every integer can be expressed as the solution of an equation in the form $a + x = b$, a and b counting numbers.

Answers to Class Exercises

1. Write $2 \cdot 2$ as $2(1 + 1)$ and use the distributive property to arrive at $2 \cdot 2 = 2 + 2$.

2. The distributive law, $a(b + c) = ab + ac$, on the left side tells us something about the sum of two counting numbers. The product $3(5 + 6 + 8)$ involves the sum of three numbers. It is still possible to use the distributive property, for $5 + 6 + 8$ means $(5 + 6) + 8$. That is, $5 + 6 + 8$ may be regarded as the sum of two numbers, one named $5 + 6$ and the other named 8. Now write $3(5 + 6 + 8)$ as $3\big((5 + 6) + 8\big)$. From the distributive property this may be written as $3(5 + 6) + 3 \cdot 8$. One other application of the distributive property gives the required result.

3. There are many ways to do this problem, all requiring the use of the properties of the counting numbers. Here is one.

$$\begin{aligned} 5 \cdot (6 \cdot 9) &= (5 \cdot 6) \cdot 9 && \text{Associative property of multiplication} \\ &= 9 \cdot (5 \cdot 6) && \text{Commutative property of multiplication} \\ &= 9 \cdot (6 \cdot 5) && \text{Commutative property of multiplication} \end{aligned}$$

4. This problem may also be done in many orders. To group 4, 5, and 6 together we think of $5 + 6$ as one number and use the associative property $(a + b) + c = a + (b + c)$. All properties used apply to addition.

$$\begin{aligned} (3 + 4) + (5 + 6) &= 3 + \big(4 + (5 + 6)\big) && \text{Associative Property} \\ &= \big(4 + (5 + 6)\big) + 3 && \text{Commutative Property} \\ &= \big(4 + (6 + 5)\big) + 3 && \text{Commutative Property} \\ &= \big((4 + 6) + 5\big) + 3 && \text{Associative Property} \\ &= \big((6 + 4) + 5\big) + 3 && \text{Commutative Property} \end{aligned}$$

5. No. For the counting numbers to have an identity with respect to addition there would have to be a counting number which added to 1 would give the sum 1. (Remember, 0 is not a counting number.) To prove something is true we must show it true in all cases. To show a general statement is not true, it is enough to show that it does not hold in one special case, as we have done here.

6. (a) $\frac{11}{3}$ (b) $\frac{15}{6}$ (c) $\frac{93}{10}$ (d) $\frac{9}{8}$

7. (a) $7x = 2$ (b) $4x = 3$ (c) $12x = 5$ (d) $100x = 90$

8. (a) yes (b) no

9. Yes. The number named by $\frac{2}{5}$ is a solution of $5x = 2$ and $\frac{40}{100}$ is a name for the solution of the equation $100x = 40$. To apply the test we compare the equations

$$100 \cdot 5x = 100 \cdot 2$$

and

$$5 \cdot 100x = 5 \cdot 40 .$$

The multipliers 100, for the first equation, and 5, for the second equation were chosen so that the left-hand sides of both test equations are equal. Thus, to decide if the two equations are the same, we need only compare the two right sides.

The answer also follows directly from the statement: $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$, b and d unequal to zero. (See answer to Exercise 10 below.)

10. Yes. The number named by $\frac{a}{b}$ is a solution of the equation $bx = a$ and $\frac{c}{d}$ is the name for the solution of the equation $dx = c$. To apply the test we compare the equations

$$dbx = da$$

and

$$bdx = bc.$$

The multipliers, d for the first equation and b for the second equation, were chosen so that the two left sides of both test equations are equal. Thus, to decide if the two equations are the same, we need only compare the two right sides. If $ad = bc$, $\frac{a}{b}$ and $\frac{c}{d}$ name the same number.

11. This question cannot be baldly answered yes or no. The symbol $\frac{4}{1}$ is a name for a solution of the equation $1 \cdot x = 4$. The equation $1 \cdot x = 4$ also has a counting number as a solution; namely, $x = 4$. We agree that these two symbols should name the same number. Our intuitive notion of fraction corroborates this agreement; $\frac{4}{1}$ths of a pie would be 4 pies.

12. Yes. The equations for $\frac{1}{3}$ and $\frac{11}{33}$ are, respectively, $3x = 1$ and $33x = 11$. To perform the test of the definition we examine the equations

$$3 \cdot 33x = 1 \cdot 33$$
$$3 \cdot 33x = 3 \cdot 11.$$

Since $33 = 3 \cdot 11$ we see that the equation $3 \cdot 33x = 1 \cdot 33$ may be obtained from $11 \cdot 3x = 11 \cdot 1$ (multiply by 3). The test of the definition requires that we multiply the equation for $\frac{11}{33}$, $33x = 11$, also by 3. Thus, by noting these facts we can assure ourselves that the test is satisfied and save some multiplications.

13. (a) $>$ (b) $>$ (c) $>$ (d) $>$

14. (a) $<$ (b) $<$ (c) $<$ (d) $<$
(e) $=$ (f) $=$ (g) $=$

15. This may be readily seen by using the test of the definition.

The equation $bkx = ak$ has the solution $x = \frac{ak}{bk}$.

The equation $bx = a$ has the solution $x = \frac{a}{b}$.

Multiplying the first by b gives

$$bbkx = bak.$$

Multiplying the second by bk gives

$$bkbx = bka.$$

16. We know from the given information that $cf < de$. It follows that $cf \cdot k < de \cdot k$ which proves the assertion.

17. To satisfy our physical intuition regarding rational numbers, we would like a segment corresponding to $\frac{1}{2}$ to be twice as long as a segment corresponding to $\frac{1}{4}$.

18. We return to the equations $2x = 0$ and $5x = 0$. Multiplying by 5 and 2, respectively, we obtain

$$5 \cdot 2x = 5 \cdot 0 = 0$$
$$2 \cdot 5x = 2 \cdot 0 = 0.$$

As these two equations are the same, $\frac{0}{2}$ and $\frac{0}{5}$ name the same rational number. Moreover, the equations $2x = 0$ and $5x = 0$ have 0 as solution so we identify 0 with $\frac{0}{2}$ and $\frac{0}{5}$.

19. Yes for mod 5. No for mod 4, since in addition to having at least one factor 0 to give a 0 product we also have

$$2 \times 2 \equiv 0 \pmod{4}.$$

20. (a) 7 (b) 3 (c) $^{-}1$ (d) $^{-}1$ (e) $^{-}6$ (f) 0

21. (a) 9 (b) 1 (c) $^{-}1$ (d) 0 (e) 0 (f) $^{-}6$

22. (a) <
(b) <
(c) >
(d) <
(e) <
(f) =
(g) <
(h) =

23. (a) (3,8) (b) (9,16) (c) (5,3) (d) (1,100)

24. (a), (c), (d)

25. (c)

26. (a), (b), (c)

Chapter 6

BINARY OPERATIONS

Introduction

In Chapter 5 the rational numbers were introduced and accommodated on the number line. In this chapter binary operations will be defined on the rational numbers and the properties of these binary operations will be investigated. When we formulate the definition of these binary operations we will want the arithmetic of rational numbers to reflect our past experiences. For example, our experience dictates that $\frac{1}{2} + \frac{1}{2}$ should be 1. In an idealized form $\frac{1}{2}$ a pie plus $\frac{1}{2}$ a pie is a pie. This is, of course, idealized; it is extremely difficult to put two halves of a cherry pie together to have a whole pie. We shall also find ourselves motivated by what we regard as desirable features of a number system.

In the last chapter we looked first at the positive rationals as the set of all solutions to equations in the form $bx = a$ where a and b are counting numbers. We then introduced zero as a rational number by considering all solutions to the equation $bx = 0$ where b is a counting number. Last, we took the opposites of all the positive rationals to form the negative rationals. These three sets, the positive rationals, the negative rationals, and zero together form the *set of rational numbers*. The counting numbers, the whole numbers, and the integers are all contained in the set of rational numbers and hence each is a subset of the set of rational numbers.

In naming these numbers we agreed to identify symbols such as 4 with a counting number and fractions such as $\frac{8}{2}$ with a rational number. Though the symbols 4 and $\frac{8}{2}$ have different geneologies, we agree that they name the same number. The words "horse" and "cheveau" have different origins but they name the same animal. A person who speaks both English and French would use the words interchangeably depending upon the situation. When we define binary operations for the rational numbers we shall want the definitions made in such a way that they agree with the known definitions for the counting numbers.

The point of view of the last chapter will also be used in this chapter. We have been assuming that we are inventing rational numbers. We have a certain amount of intuition to guide us and to suggest the final form of our invention. Our knowledge of rational numbers is that gained from taking them as solutions of equations. To proceed, then, we will make extensive use of this defining knowledge. This point of view is different from that given in most texts. For example, in *Mathematics for Junior High School*, Vol. 1, it is assumed that there are rational numbers, that binary operations are defined on them, and that these binary operations have certain properties. In this text, for teachers, we prefer to show that it is not necessary to make these assumptions since they can be shown to follow directly from the definitions of the operations.

6.1 Addition

Let us begin with an introduction of a binary operation, addition. We start with some simple specific cases to illustrate the method we will use. Suppose we wish to find a rational number to be called the sum of $\frac{1}{2}$ and $\frac{1}{2}$. To mathematically motivate this sum we return to our meaning of $\frac{1}{2}$. We think of $\frac{1}{2}$ as a solution of the equation $2x = 1$; that is, $\frac{1}{2}$ has the property that $2 \cdot \frac{1}{2} = 1$. The symbols 1 and $2 \cdot \frac{1}{2}$ are names of the same counting number. As we wish to define $\frac{1}{2} + \frac{1}{2}$, let us try to involve $\frac{1}{2}$ and $\frac{1}{2}$ in a single statement. One way to do this is to write the true statement

$$2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 1 + 1 .$$

This statement is true since $2 \cdot \frac{1}{2}$ is another name for 1. Thus, it is meaningful to write $2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2}$ since it is another name for the sum of the counting numbers 1 and 1. In our treatment as yet, we do not have a meaning attached to the sum $\frac{1}{2} + \frac{1}{2}$. If it is possible to define operations on the rational numbers such that the distributive law holds, then we would be able to obtain from

$$2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = 2$$

the statement

$$2 \cdot (\frac{1}{2} + \frac{1}{2}) = 2.$$

If the distributive law is to hold and $\frac{1}{2} + \frac{1}{2}$ is to have a meaning, we must agree that $\frac{1}{2} + \frac{1}{2}$ is the name of a solution of the equation

$$2y = 2 .$$

But we know that the equation $2y = 2$ has $y = \frac{2}{2}$ as a solution. Hence, we shall agree that $\frac{1}{2} + \frac{1}{2}$ and $\frac{2}{2}$ name the same number. Since $\frac{2}{2}$ is a name for the number one, we define the sum $\frac{1}{2} + \frac{1}{2}$ to be 1.

Let us go through this in another simple case. The rational number $\frac{1}{3}$ is a solution of the equation $3x = 1$, or equivalently, $3 \cdot \frac{1}{3} = 1$. To motivate a meaning for $\frac{1}{3} + \frac{1}{3}$, we proceed as before. The definition of $\frac{1}{3}$

tells us that $3 \cdot \frac{1}{3} = 1$. To relate $\frac{1}{3}$ and $\frac{1}{3}$ we write the true statement

$$3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} = 1 + 1$$

or

$$3(\frac{1}{3} + \frac{1}{3}) = 2 .$$

This tells us that if a distributive law holds we want to call $\frac{1}{3} + \frac{1}{3}$ a solution of the equation $3z = 2$. That is, the sum $\frac{1}{3} + \frac{1}{3}$ should be called $\frac{2}{3}$ as we know $\frac{2}{3}$ is a name for the solution of $3z = 2$.

The two examples above indicate the procedure that shall be used to define addition of rational numbers. Clearly, they were very specialized examples and examples for which the decisions could easily have been made from physical models. Now let us look at something which is less obvious physically. To define the sum of $\frac{2}{3}$ and $\frac{5}{8}$ we may begin as before. The number $\frac{2}{3}$ is a solution of $3x = 2$ and $\frac{5}{8}$ is a solution of $8y = 5$; that is, $3 \cdot \frac{2}{3} = 2$ and $8 \cdot \frac{5}{8} = 5$. To combine $\frac{2}{3}$ and $\frac{5}{8}$ we may try the above method; combine the two equations:

$$3 \cdot \frac{2}{3} + 8 \cdot \frac{5}{8} = 2 + 5 = 7.$$

This time, however, there is a difference. Even with the use of the distributive law we are unable to group together $\frac{2}{3}$ and $\frac{5}{8}$.

What to do? The first step may be to ask why the procedure failed. To answer this question we must be clear on what the procedure was. To go from the statement $3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} = 2$ to the statement $3(\frac{1}{3} + \frac{1}{3}) = 2$ required the use of distributivity. In general, the distributive property is stated as

$$ab + ac = a(b + c).$$

The expression $3 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3}$ seems tailor-made to use a distributive property as we have a common factor (multiplier). This is, of course, the reason for the difficulty with

$$3 \cdot \frac{2}{3} + 8 \cdot \frac{5}{8} ;$$

there is no common factor!

Should this technique be abandoned? This question is important and deserves some serious thought before an answer is given. One line of thought might lead us back to the preceding chapter and the introduction of rational numbers and fractions. On several occasions we wished to compare numbers named by fractions. This was done in such a way that the new equations had equal co-efficients.

After this reflection let us return to our problem, and see if this train of thought has been useful. The fraction $\frac{2}{3}$ is the solution of $3x = 2$, so that $3 \cdot \frac{2}{3} = 2$. Also $\frac{5}{8}$ has the property $8 \cdot \frac{5}{8} = 5$. Let us multiply the first equation by 8 and the second equation by 3. (The multipliers are the denominators of the two fractions involved.) These multiplications yield

$$8 \cdot (3 \cdot \frac{2}{3}) = 16$$

$$3 \cdot (8 \cdot \frac{5}{8}) = 15$$

or assuming the associative property,

$$(8 \cdot 3) \cdot \frac{2}{3} = 16$$

$$(3 \cdot 8) \cdot \frac{5}{8} = 15.$$

Now let us add the counting numbers 16 and 15 and multiply the counting numbers 8 and 3.

$$(8 \cdot 3) \cdot \frac{2}{3} + (3 \cdot 8) \cdot \frac{5}{8} = 16 + 15$$

$$24 \cdot \frac{2}{3} + 24 \cdot \frac{5}{8} = 31.$$

Using the distributive property, we get

$$24(\frac{2}{3} + \frac{5}{8}) = 31.$$

Thus, it is seen that $\frac{2}{3} + \frac{5}{8}$ should be a name for the solution of the equation $24z = 31$ and that we should say

$$\frac{2}{3} + \frac{5}{8} = \frac{31}{24} .$$

The general case for the sum of any two rational numbers is treated in a similar fashion. Let $\frac{a}{b}$ and $\frac{c}{d}$ name two rational numbers. These rational numbers are the solutions of the equations $bx = a$ and $dy = c$, respectively. By this is meant $b \cdot \frac{a}{b} = a$ and $d \cdot \frac{c}{d} = c$. Equivalently we have

$$d(b \cdot \frac{a}{b}) = da \qquad \text{and} \qquad b(d \cdot \frac{c}{d}) = bc.$$

Assuming the associative property, we can combine to get

$$(bd)\frac{a}{b} + (bd)\frac{c}{d} = ad + bc.$$

Note that the commutative property for the multiplication of whole numbers has been used to write db as bd and da as ad.

Using the distributive property gives

$$bd(\frac{a}{b} + \frac{c}{d}) = ad + bc.$$

Thus, if the operation of addition is to be extended in a natural way to rational numbers, we would want to say that $\frac{a}{b} + \frac{c}{d}$ is a solution of the equation

$$(bd)z = ad + bc.$$

This would lead us to conclude that

$$\frac{a}{b} + \frac{c}{d} \quad \text{and} \quad \frac{ad + bc}{bd}$$

name the same number.

Our thinking has led us to a plausible meaning for $\frac{a}{b} + \frac{c}{d}$. We have not offered a proof but rather an extended development to motivate a definition. Having arrived at this point, we can now wipe the slate clean and begin with the following:

Definition: The sum of any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ad + bc}{bd}$.

We have given a lengthy introduction to a relatively simple definition. There are several reasons for this verbosity. By doing this very slowly we hoped to convince the reader that addition of rational numbers is the work of man and that the definition was not deliberately designed to be as difficult as possible. The definition was arrived at through a review of the meaning of rational number and a desire to create binary operations which have properties we have found useful when working with addition and multiplication of whole numbers. It remains, of course, to be shown that this binary operation so defined does have these familiar properties, such as commutativity and associativity. In the next section we shall study the properties.

Many texts including Mathematics for Junior High School, Vol. 1, suggest that we add $\frac{2}{3}$ and $\frac{5}{8}$ by finding a common denominator. That is, one would write:

$$\begin{aligned} \frac{2}{3} + \frac{5}{8} &= \frac{2}{3} \cdot \frac{8}{8} + \frac{5}{8} \cdot \frac{3}{3} \\ &= \frac{16}{24} + \frac{15}{24} \\ &= \frac{1}{24}(16 + 15) \\ &= \frac{1}{24}(31) \\ &= \frac{31}{24}. \end{aligned}$$

The treatment in this text appears to be greatly different. However, the difference is more a philosophical difference than a mechanical difference. It will be seen that the mechanics of the two methods are really the same. The treatment in this text has been different to emphasize to the teacher that addition of rational numbers may be motivated and finally accomplished without multiplying fractions. The discussion of addition has depended upon the whole numbers. That is, we have made addition of rationals relate to addition and multiplication of whole numbers.

To see the similarity of the two methods, let us review the method of this text. To collect together $\frac{2}{3}$ and $\frac{5}{8}$ we multiply the equations

$$3 \cdot \frac{2}{3} = 2 \quad \text{and} \quad 8 \cdot \frac{5}{8} = 5$$

by 8 and 3, respectively. This gives us

$$8 \cdot (3 \cdot \frac{2}{3}) = 8 \cdot 2 \quad \text{and} \quad 3 \cdot (8 \cdot \frac{5}{8}) = 3 \cdot 5.$$

The first equation is of the form $8 \cdot 3x = 8 \cdot 2$ which has a solution named by $\frac{8 \cdot 2}{8 \cdot 3}$. Hence, $\frac{8 \cdot 2}{8 \cdot 3} = \frac{2}{3}$. The second equation is of the form $3 \cdot 8x = 3 \cdot 5$ which has a solution named by $\frac{3 \cdot 5}{3 \cdot 8}$. Hence, $\frac{3 \cdot 5}{3 \cdot 8} = \frac{5}{8}$. We see that we have done the same work in both methods. Only the style is different.

Seventh grade texts generally introduce multiplication before addition, the reasons being that multiplication seems simpler than addition and that multiplication may be used in the computation of sums. Note, however, that both treatments use properties either to motivate the discussion or to carry out the computation.

Example: Use the definition of the sum of two rational numbers to find the sum $\frac{5}{18} + \frac{3}{8}$.

From the definition we have

$$\frac{5}{18} + \frac{3}{8} = \frac{5 \cdot 8 + 18 \cdot 3}{18 \cdot 8}$$

which may be written as

$$\frac{40 + 54}{144} = \frac{94}{144}.$$

The fraction $\frac{94}{144}$ names a rational number which has many names. At this stage we will not want to find the "simplest" name.

Class Exercises

1. Use the definition to find the sums:

   (a) $\frac{2}{3} + \frac{3}{5}$ (c) $\frac{1}{2} + \frac{1}{2}$ (e) $\frac{1}{3} + \frac{1}{6}$

   (b) $\frac{5}{2} + \frac{1}{8}$ (d) $\frac{1}{3} + \frac{1}{3}$ (f) $\frac{2}{4} + \frac{1}{2}$

2. The form of the answers to (c) and (d) of problem 1 will not be the same as the form of the sums obtained in the text. Are the sums themselves different?

3. (a) Use the definition to find the sum $\frac{4}{1} + \frac{3}{1}$.

   (b) Does the answer to part (a) agree with the fact that $\frac{4}{1}$ and $\frac{3}{1}$ are fractional names for 4 and 3 ?

4. (a) Use the definition to find the sum $\frac{28}{7} + \frac{10}{2}$.

   (b) Does the answer to part (a) agree with the fact that $\frac{28}{7}$ and $\frac{10}{2}$ are fractional names for 4 and 5, respectively.

6.2 Properties of Addition

The binary operation of addition on the rational numbers has been introduced and defined. Now is the time to investigate this operation to show that it does have the desired properties similar to addition on the whole numbers. We repeat that the sum of $\frac{a}{b}$ and $\frac{c}{d}$, $b, d, \neq 0$, is defined to be $\frac{ad + bc}{bd}$; the sum of $\frac{a}{b}$ and $\frac{c}{d}$ is the solution of the equation $bdx = ad + bc$. Since a, b, c, and d are whole numbers, so are bd and $ad + bc$. Thus, $\frac{a}{b} + \frac{c}{d}$ is the solution of an equation stated with whole numbers, $bd \neq 0$, which means $\frac{a}{b} + \frac{c}{d}$ is the name of a rational number. We have proved that the binary operation introduced in the last section is closed; the sum of two rational numbers is a rational number.

What else can we say about this binary operation? Let us compare $\frac{3}{4} + \frac{9}{10}$ and $\frac{9}{10} + \frac{3}{4}$. By the definition of addition, $\frac{3}{4} + \frac{9}{10}$ is

$$\frac{3 \cdot 10 + 4 \cdot 9}{4 \cdot 10}$$

and is the solution of the equation $4 \cdot 10x = 3 \cdot 10 + 4 \cdot 9$. By the definition of addition, $\frac{9}{10} + \frac{3}{4}$ is

$$\frac{9 \cdot 4 + 10 \cdot 3}{10 \cdot 4}$$

and is the solution of the equation $10 \cdot 4x = 9 \cdot 4 + 10 \cdot 3$. The equations are stated in terms of whole numbers as are the results of the definition. However, for the whole numbers multiplication and addition are commutative. Hence, we can show that the second result equals the first:

$$\frac{9 \cdot 4 + 10 \cdot 3}{10 \cdot 4} = \frac{4 \cdot 9 + 3 \cdot 10}{4 \cdot 10}$$

(commutative property for multiplication of whole numbers)

$$= \frac{3 \cdot 10 + 4 \cdot 9}{4 \cdot 10} .$$

(commutative property for addition of whole numbers)

Likewise, using the same properties we can show that the two equations are identical. We therefore conclude that

$$\frac{3}{4} + \frac{9}{10} = \frac{9}{10} + \frac{3}{4} .$$

The method that was used here will work in general to show that addition of rational numbers is commutative. For the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$

$$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b} .$$

That we can prove the commutative property holds for rational numbers is a reflection of the fact that the treatment of rationals in this text is "deeper" than that given in a junior high school text. We repeat once again: We don't expect the teacher to present this development to her classes but the teacher should see an orderly development of rational numbers.

The associative property may also be seen to hold for addition of rational numbers by following the definition through in much the same way as was done with the commutative property. In this case, however, the proof rests upon the use of the associative property for addition of whole numbers. When stated in terms of rational numbers $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$, the associative property becomes

$$\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right).$$

We have decided that the fractions such as $\frac{0}{1}$, $\frac{0}{2}$, and $\frac{0}{17}$, are different names for 0. Let us see how 0 behaves with respect to addition as we have defined this operation. To find the sum of $\frac{0}{1}$ and $\frac{3}{17}$ we use the definition:

$$\frac{0}{1} + \frac{3}{17} = \frac{0 \cdot 17 + 1 \cdot 3}{1 \cdot 17} = \frac{3}{17} .$$

That is, the rational number $\frac{0}{1}$ acts as an additive identity. (We have examined the behavior of $\frac{0}{1}$ only when combined with $\frac{3}{17}$ but it is clear, is it not, that the pattern would be the same with any rational number.)

Again we have not proved a general statement. Rather an indication has been given that 0 is an additive identity. For any rational number $\frac{a}{b}$, $b \neq 0$, it is true that $0 + \frac{a}{b} = \frac{a}{b}$. This should be interpreted as being true regardless of which name we use for 0. The reader should try a few examples. Is it true that $\frac{0}{1} + \frac{9}{11} = \frac{9}{11}$? Do the two sides of the equation $\frac{0}{23} + \frac{9}{11} = \frac{9}{11}$ name the same number?

We have seen that by carefully inventing the rational numbers and a binary operation on them we have a mathematical system with properties that are familiar to us.

Before leaving addition there is another matter which needs comment. It has been agreed that we will name a whole number with certain fraction names. Since 4 and $\frac{4}{1}$ solve the same equation we have agreed that 4 and $\frac{4}{1}$ name the same number. An addition for numbers with fraction names has just been described. Thus, given two whole numbers, say 4 and 3, we have two ways to perform addition. We may write $4 + 3 = 7$ or we may do the addition using fractions. The fractions $\frac{4}{1}$ and $\frac{3}{1}$ name the same numbers as 4 and 3:

$$\frac{4}{1} + \frac{3}{1} = \frac{4 \cdot 1 + 1 \cdot 3}{1 \cdot 1} = \frac{7}{1}.$$

Fortunately, $\frac{7}{1}$ and 7 name the same number. A grain of sand does not make a mountain, nor does one example prove a general statement. In this situation the one example does, however, give an insight into the general case.

The one example and the general statement which may be proved similarly tells us that the addition introduced on the rational numbers is an extension of the addition we know for whole numbers. This is highly desirable. We have two ways to add whole numbers; one is essentially finger counting, and the other is to rename the whole numbers as fractions and use addition of rational numbers. Had we obtained different answers for $4 + 3$ and $\frac{4}{1} + \frac{3}{1}$ our intuitive concept could not hold.

Class Exercises

5. Use the definition to perform the additions in (a), (b), (c), and (d).

(a) $\frac{3}{4} + \frac{17}{29}$

(b) $\frac{17}{29} + \frac{3}{4}$

(c) $\frac{0}{101} + \frac{6}{5}$

(d) $\frac{6}{5} + \frac{0}{101}$

(e) Compare the answers to (a) and (b).

(f) Compare the answers to (c) and (d).

6. (a) Express as a fraction the sum $\frac{3}{4} + \frac{2}{3}$.

(b) Use the answer to (a) to put $(\frac{3}{4} + \frac{2}{3}) + \frac{6}{10}$ in fractional form.

(c) Put $\frac{2}{3} + \frac{6}{10}$ in fractional form.

(d) Use the answer to (c) to put $\frac{3}{4} + (\frac{2}{3} + \frac{6}{10})$ in fractional form.

(e) Compare the answers to (b) and (d).

7. Put $\frac{18}{3} + \frac{25}{5}$ in fractional form using the definition of addition. Do the corresponding addition using non fractional names for these rational numbers. Compare your answers.

8. Students like to think that addition of $\frac{a}{b}$ and $\frac{c}{d}$ should be defined as

$$\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}.$$

Why is this a most unsatisfactory definition?

Hint: Use the definition to find the sum $\frac{1}{2} + \frac{1}{2}$.

What should a teacher teach to a seventh grade class about rational numbers? Certainly, that our definition for the addition of rationals is reasonable. That is, the operation we call addition is defined exactly as our reason tells us it should be. Students tend to view rational numbers and operations on them as mysterious. This is particularly true for addition. The judicious use of equations can dispel much of this mystery. The properties of addition and of 0 should be stressed. The notion that 4 and $\frac{8}{2}$ must be identified as two names for the same number is probably too subtle at this point. However, the student will be willing to accept, without any discussion, that 4 and $\frac{8}{2}$ are two names for the same number.

A teacher who is confident working with rational numbers will be able to instill this confidence to the class. Operating on the rational numbers is not difficult. As has been seen, it depends only on a good working knowledge of the whole numbers. Class Exercise 8 may be used to discourage one prevalent false idea, particularly if illustrated with half dollars.

Students generally regard multiplication of rational numbers as simpler than addition. There are probably several reasons for this. To begin with, addition is introduced through the use of multiplication. Secondly, as teacher, we generally want our students to be efficient and use the lowest common denominator when adding. We frequently mark an answer wrong simply because it is not reduced. To illustrate this, take the problem of putting $\frac{5}{36} + \frac{7}{30}$ in fractional form. Following our method the sum would be $\frac{402}{1080}$. We motivated this by multiplying the equation $36x = 5$ and $30y = 7$ by 30 and 36, respectively, to obtain in each case the coefficient 1080. The common coefficient suggests the distributive law. We could also have obtained a common coefficient of 180 by multiplying the equations by 5 and 6, respectively, to obtain $5 \cdot 36x = 5 \cdot 5$ and $6 \cdot 30y = 6 \cdot 7$ or $180x = 25$ and $180y = 42$; thus, $180(x + y) = 67$. It is true that our second answer appears simpler and that we prefer the answer $\frac{67}{180}$. This is not mathematical reasoning but psychological. We must remember that it is better to get a correct answer rather than to worry too much about efficiency.

The essence of addition as usually taught in the elementary grades is finding a common denominator. When we insist, at an early stage, that the student find the least common denominator, the student may lose sight of the meaning of addition. The student should thoroughly learn that rational numbers have many names. The most useful name will depend on the circumstances.

6.3 Multiplication

Now that addition of rational numbers has been introduced, we wish to introduce a second binary operation. The operation of addition was motivated through the properties and operations of the whole numbers. The second binary operation, multiplication, will also be motivated through the whole numbers.

Let us look first at an example. The rational number $\frac{3}{4}$ is the solution of the equation $4x = 3$; $4 \cdot \frac{3}{4} = 3$. Also, $\frac{7}{5}$ is the solution of the equation $5y = 7$; $5 \cdot \frac{7}{5} = 5$. The numbers 3 and 7 have a well determined product, $3 \cdot 7 = 21$. As $4 \cdot \frac{3}{4}$ and $5 \cdot \frac{7}{5}$ are other names for 3 and 7

they, too, have a well determined product. Thus, it is meaningful to write

$$(4 \cdot \frac{3}{4}) \cdot (5 \cdot \frac{7}{5}) = 3 \cdot 7 .$$

Remember we are merely exploring, not proving, and so may use a bit of sleight of hand to rewrite this equation as

$$(4 \cdot 5) \cdot (\frac{3}{4} \cdot \frac{7}{5}) = 3 \cdot 7.$$

(Here we have proceeded as if it is meaningful to use associativity and commutativity for the operation of multiplication with rational numbers.)

The displayed equation above does suggest to us that $\frac{3}{4} \cdot \frac{7}{5}$ should be the solution of

$$(4 \cdot 5)z = 3 \cdot 7.$$

The solution of this equation is named $\frac{3 \cdot 7}{4 \cdot 5}$. Hence, it seems reasonable to say

$$\frac{3}{4} \cdot \frac{7}{5} = \frac{3 \cdot 7}{4 \cdot 5} = \frac{21}{20} .$$

Definition: The product of any two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{ac}{bd}$.

This definition may be rephrased as: The product of two rational numbers written as fractions may be represented as a fraction whose numerator is the product of the numerators and whose denominator is the product of the denominators.

Once again the reader has observed a slightly different approach to the treatment of rational numbers. The reason for this approach is as before, to emphasize the equation meaning of rational numbers. The rational numbers are known through equations and the equations have been used to motivate the definitions. We also use the properties that we would like addition and multiplication to possess to suggest these operations to us. Having arrived at what seem to be reasonable ideas of addition and multiplication, we then show that the properties hold. Other treatments take the properties for granted and show that the definitions given must be the definitions used. The net result is, of course, the same operation.

The slight differences in introducing rational numbers are not as important as the similarity in the newer texts. This similarity is a pedagogical similarity. Rather than present the arithmetic of rationals as an irrevocable law of nature, which we must all unthinkingly obey, arithmetic is presented as an organized, conscious development of man to suit his purposes. Children sometimes ask questions about mathematics that seem naive but are really penetrating. "Who decided $1 + 1 = 2$?," or "Who decided that $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$?" These and other questions have answers when mathematics is developed rather than merely presented as a fact.

Class Exercises

9. Use the definition of multiplication to find the product $\frac{4}{1} \cdot \frac{3}{1}$. Does the answer agree with the fact that $\frac{4}{1}$ and $\frac{3}{1}$ are fractional names for 4 and 3 ?

10. Use the definition to find the product $\frac{28}{7} \cdot \frac{10}{2}$. Does this answer agree with the fact that $\frac{28}{7}$ and $\frac{10}{2}$ are fractional names for counting numbers?

6.4 Properties of Multiplication

We would like to show that multiplication of rational numbers, as defined, is as well behaved as addition. That is, we would like to show that multiplication is closed, commutative, and associative. Furthermore, we like to know that there is an identity with respect to multiplication and that multiplication of rationals is an extension of multiplication of whole numbers.

For whole numbers a, b, c, and d with $b, d \neq 0$, the product of the rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is defined to be $\frac{ac}{bd}$. The symbol $\frac{ac}{bd}$ names the solution of the equation $(bd)x = ac$. Thus, the product of two rational numbers is again a rational number. The binary operation of multiplication is closed.

Do not expect a seventh grader to turn cartwheels in the aisle as you announce this fact. His reaction is apt to be the bored "So what," or the pseudo sophisticated "Of course." It may well be beneficial to repeat some examples of binary operations which are not closed: Subtraction on the set of counting numbers, division on the set of whole numbers, or multiplication on the set of numbers 1, 2, 3, 4.

Is multiplication commutative? For rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ this asks: Is it true that $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$? We have $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ and $\frac{c}{d} \cdot \frac{a}{b} = \frac{ca}{db}$. Do the two fractions $\frac{ac}{bd}$ and $\frac{ca}{db}$ name the same number? Using the commutative property for multiplication of whole numbers the numerator and denominator of the second fraction can be shown to equal those of the first. Hence, the multiplication of rational numbers is *commutative*.

Given three rational numbers it is not difficult to show that the *associative property* holds for multiplication. Indeed, it is no more difficult to show the associative property in general than in a special case. Some examples will convince the reader of this.

The number 1 has a special property with respect to multiplication of whole numbers. Does this property extend to the rational numbers? Let us look at an example. To find the product of 1 and $\frac{4}{7}$ we must first rename 1 with a fractional name. There are many names to choose from; suppose we use $\frac{3}{3}$. The product $\frac{3}{3} \cdot \frac{4}{7} = \frac{3 \cdot 4}{3 \cdot 7} = \frac{12}{21}$. The number named by $\frac{12}{21}$ is the same number as named by $\frac{4}{7}$.

To show that 1 is a multiplicative identity we use the name $\frac{k}{k}$, $k \neq 0$, for 1 and take an arbitrary rational number $\frac{a}{b}$. Now $\frac{k}{k} \cdot \frac{a}{b} = \frac{ka}{kb}$. In Class Exercise 15 of Chapter 5 it was shown that $\frac{ka}{kb}$ and $\frac{a}{b}$ name the same number. We frequently write this as $1 \cdot \frac{a}{b} = \frac{a}{b}$. Thus, we have proved that 1 is the *multiplicative identity*.

The whole numbers have been identified with certain rational numbers. When we say that the whole numbers are a subset of the rational numbers, we mean that there is a subset of the rational numbers that solve the same equations as do the whole numbers. Now an operation called multiplication has been introduced on the rational numbers. Seemingly then, there are two forms of multiplication for whole numbers: The multiplication as learned for whole numbers, and the multiplication forced on the whole numbers when they are regarded as a subset of the rationals. As an example there is the product $9 \cdot 7 = 63$. We also may write 9 as $\frac{18}{2}$ and 7 as $\frac{35}{5}$, then the product $\frac{18}{2} \cdot \frac{35}{5}$ equals $\frac{630}{10}$, which is easily seen to be another name for $\frac{63}{1}$ or to be identified with 63. This is true in general; the product of two whole numbers determined with fractions or with decimal numerals is the same. This is much like two students who work the same arithmetic problem. If one student uses blue ink and another uses black, the external form will be different but the arithmetical result will be the same.

Class Exercises

11. Show by computing that

$$\frac{3}{4} \cdot \left(\frac{9}{7} \cdot \frac{8}{11}\right) = \left(\frac{3}{4} \cdot \frac{9}{7}\right) \cdot \frac{8}{11} .$$

12. Show by computation that

$$\frac{3}{4} \cdot \left(\frac{9}{7} + \frac{8}{11}\right) = \left(\frac{3}{4} \cdot \frac{9}{7}\right) + \left(\frac{3}{4} \cdot \frac{8}{11}\right).$$

13. Determine the rational number which is a solution of the equation

$$\frac{3}{8} \cdot x = \frac{7}{5} .$$

14. Evaluate:

(a) $\frac{3}{4} \cdot \frac{4}{3}$

(b) $\frac{9}{10} \cdot \frac{10}{9}$

(c) $\frac{2}{5} \cdot \frac{5}{2}$

(d) $\frac{9}{7} \cdot \frac{7}{9}$

15. Is it meaningful to write $1 = \frac{96}{96}$? If so, in what respect is it meaningful?

6.5 The Distributive Property

Problem 12 of the preceding set of Class Exercises is an example of the working of the distributive property. For rational numbers the distributive property states:

> For a, b, c, d, e, and f whole numbers with b, d, f, $\neq 0$, it is true that
>
> $$\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} \cdot \frac{c}{d}\right) + \left(\frac{a}{b} \cdot \frac{e}{f}\right).$$

It is a straightforward approach to show that the property holds. Simply compute both sides and see that they name the same rational number. The left side can be expressed as follows:

$$\begin{aligned}\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) &= \frac{a}{b} \cdot \left(\frac{cf + de}{df}\right) \\ &= \frac{a(cf + de)}{b(df)} \\ &= \frac{a(cf) + a(de)}{b(df)} .\end{aligned}$$

The last step uses the distributive property for whole numbers. The right side can be expressed as follows:

$$\left(\frac{a}{b}\cdot\frac{c}{d}\right)+\left(\frac{a}{b}\cdot\frac{e}{f}\right)=\frac{ac}{bd}+\frac{ae}{bf}$$

$$=\frac{(ac)\cdot(bf)+(bd)\cdot(ae)}{(bd)\cdot(bf)}\ .$$

The two fractions $\frac{a(cf)+a(de)}{b(df)}$ and $\frac{(ac)\cdot(bf)+(bd)\cdot(ae)}{(bd)\cdot(bf)}$ are quite different in appearance. Do they name the same number? Using the properties of whole numbers the two fractions may be rewritten as

$$\frac{acf+ade}{bdf} \quad \text{and} \quad \frac{b(acf+ade)}{b(bdf)}\ .$$

From Class Exercise 15 of Chapter 5 it follows that the two fractions name the same rational number. Thus, it has been shown that the distributive law is valid.

6.6 Subtraction

With the introduction of addition and multiplication it is possible to introduce subtraction and division. Subtraction and division are not to be regarded as new operations. It shall be shown that addition and multiplication can be used to solve problems that are usually considered as subtraction and division problems.

We may determine by the operation addition a rational number we call

$$\frac{3}{4}+\frac{7}{5}\ .$$

To determine the rational number we call $\frac{2}{3}-\frac{4}{7}$, we may proceed in a routine way:

$$\frac{2}{3}-\frac{4}{7}=\frac{2}{3}\cdot\frac{7}{7}-\frac{4}{7}\cdot\frac{3}{3}$$

$$=\frac{14}{21}-\frac{12}{21}$$

$$=\frac{1}{21}(14-12)$$

$$=\frac{1}{21}\cdot 2$$

$$=\frac{2}{21}\ .$$

(In the third step we have made use of a distributive property of multiplication over subtraction.)

Let us look deeper into the meaning of the rational number we call $\frac{2}{3} - \frac{4}{7}$ by use of the equation method. When we evaluate $\frac{2}{3} - \frac{4}{7}$, we determine a value for x such that $\frac{4}{7} + x = \frac{2}{3}$. This is comparable to saying that the solution of the equation $9 + n = 15$ is $15 - 9$. To see more clearly the relation of subtraction to addition let us use the equation $\frac{4}{7} + x = \frac{2}{3}$ to find $\frac{2}{3} - \frac{4}{7}$.

When we attempt to solve this equation we are asking: Is there a rational number which may be substituted for x in

$$\frac{4}{7} + x = \frac{2}{3}$$

to make a true statement? To have something to talk about, let us think of x as a rational number, $\frac{u}{v}$. We wish to determine whole number replacements for u and v such that

$$\frac{4}{7} + \frac{u}{v} = \frac{2}{3} .$$

If there are such numbers, we may add $\frac{4}{7}$ and $\frac{u}{v}$ to obtain $\frac{4v + 7u}{7v} = \frac{2}{3}$. These two fractions, $\frac{4v + 7u}{7v}$ and $\frac{2}{3}$, will name the same number if

$$3(4v + 7u) = 2 \cdot 7v ,$$

by the definition of equivalent fractions. To see if it is possible to determine values for u and v so that this equation holds, let us rewrite it:

$$3(4v + 7u) = 2 \cdot 7v$$
$$12v + 21u = 14v$$
$$21u = 14v - 12v$$

which by a distributive property becomes

$$21u = (14 - 12)v$$
$$21u = 2v .$$

It now seems clear that we should try $v = 21$ and $u = 2$. At least these values for u and v will make the statement $21u = 2v$ a true statement. Thus, it has been suggested that $x = \frac{2}{21}$. Let us see if this works: The sum $\frac{4}{7} + \frac{2}{21}$ is $\frac{98}{147}$ which can easily be shown to name the same number as $\frac{2}{3}$.

This technique for subtraction depends only on a knowledge of addition and the meaning of fractions. A more routine technique for solving such problems may readily be presented. This method depends upon the statement: For any counting number k, the fractions $\frac{a}{b}$ and $\frac{ak}{bk}$ name the same number.

To determine $\frac{u}{v}$ such that

$$\frac{4}{7} + \frac{u}{v} = \frac{2}{3}$$

we first rename $\frac{4}{7}$ and $\frac{2}{3}$ so that they have the same denominator. Since $7 \cdot 3 = 3 \cdot 7$ we write $\frac{4}{7} = \frac{4 \cdot 3}{7 \cdot 3}$ and $\frac{2}{3} = \frac{2 \cdot 7}{3 \cdot 7}$. The equation we wish to solve may be written as

$$\frac{12}{21} + \frac{u}{v} = \frac{14}{21}.$$

It is clear that equality will hold if $\frac{u}{v}$ is $\frac{2}{21}$. This latter method, determining $\frac{u}{v}$ such that $\frac{12}{21} + \frac{u}{v} = \frac{14}{21}$, is really subtraction as taught in seventh grade. We wish to find $\frac{2}{3} - \frac{4}{7}$. The fractions $\frac{2}{3}$ and $\frac{4}{7}$ are renamed as:

$$\frac{2}{3} = \frac{2}{3} \cdot \frac{7}{7} = \frac{14}{21}$$

and

$$\frac{4}{7} = \frac{4}{7} \cdot \frac{3}{3} = \frac{12}{21}.$$

Hence,

$$\frac{2}{3} - \frac{4}{7} = \frac{14}{21} - \frac{12}{21} = \frac{2}{21}.$$

Class Exercises

16. Solve each of the equations by the two methods given above.

    (a) $\frac{9}{17} + x = \frac{2}{4}$

    (b) $\frac{2}{3} + x = \frac{4}{5}$

    (c) $\frac{9}{17} + x = \frac{9}{17}$

    (d) $\frac{99}{23} + y = \frac{99}{23}$

17. Is there any similarity between the two methods of subtraction described in this section?

The second method given above is undoubtedly the preferred method to use in the seventh grade. The first method has the advantage of stressing our basic information about rationals.

When the whole numbers are used to construct the non-negative rational numbers, we may easily write equations that do not have solutions in the set of non-negative rationals. For example, consider the equation:

$$\frac{2}{3} + x = \frac{4}{7}.$$

There is a rational number solution to this equation but it is the negative rational number, $\frac{-2}{21}$. Problems of this type can be solved using the methods previously described. However, they require familiarity with the fundamental operations on integers. These will be summarized in the last section of this chapter. We should, however, keep in mind that the procedures illustrated thus far with non-negative rational numbers may be easily extended to include all rational numbers, positive, negative, and zero.

6.7 Division

Division of whole numbers was introduced through its relationship to the multiplication of whole numbers. Corresponding to the multiplication $5 \cdot 2 = 10$ we have the divisions $10 \div 2 = 5$ and $10 \div 5 = 2$. Corresponding to the multiplication $3 \cdot n = 15$ we have the divisions $15 \div n = 3$ and $15 \div 3 = n$. Hence, to find the value of n such that $3 \cdot n = 15$ we may name n by $15 \div 3$ or $\frac{15}{3}$.

We shall also use multiplication as the basic operation in introducing division of rational numbers. To solve an equation of the form $\frac{a}{b} \cdot x = \frac{c}{d}$ we note that the result x can be expressed as $\frac{c}{d}\Big/\frac{a}{b}$ or as $\frac{c}{d} \div \frac{a}{b}$. How is this result to be evaluated and is it a rational number?

To help answer these questions let us begin the discussion with a numerical example, say

$$\frac{3}{4} \cdot x = \frac{7}{5}.$$

As we hope to find a rational number to replace x let us think of x as $\frac{u}{v}$. We want to determine the whole number replacements for u and v, $v \neq 0$, in such a way that

$$\frac{3}{4} \cdot \frac{u}{v} = \frac{7}{5}.$$

By use of the definition of multiplication this equation may be written as

$$\frac{3u}{4v} = \frac{7}{5}.$$

Here we wish to determine u and v such that $\frac{3u}{4v}$ and $\frac{7}{5}$ name the same number since this is what the last equation means. By the criterion agreed upon in Chapter 5, $\frac{a}{b}$ and $\frac{c}{d}$ will name the same number if and only if $ad = bc$. Hence, for our example, $\frac{3u}{4v}$ and $\frac{7}{5}$ will name the same number if and only if

$$(3u) \cdot 5 = (4v) \cdot 7.$$

Using the commutative property this can be expressed as

$$5 \cdot (3u) = 7 \cdot (4v) .$$

Can u and v be determined so that this last equation will be a true statement? There are many ways this can be done. The simplest way is to observe that on the left 3 and 5 appear as factors. To balance the equation let us make 3 and 5 appear as factors on the right side. This may be accomplished by choosing $v = 3 \cdot 5$.

$$5 \cdot (3u) = (7 \cdot 4) \cdot (3 \cdot 5).$$

Now to choose u so that equality holds, we assign u the value $7 \cdot 4$.

While other values of v and u would also have made the original equation true, we have chosen $v = 3 \cdot 5$ and $u = 7 \cdot 4$. That is, we have decided to choose x such that

$$x = \frac{u}{v} = \frac{7 \cdot 4}{3 \cdot 5} .$$

Let us go back to $\frac{3}{4} \cdot x = \frac{7}{5}$ and see if this replacement for x works as expected. Is it true that

$$\frac{3}{4} \cdot \frac{7 \cdot 4}{3 \cdot 5} = \frac{7}{5} \,?$$

Do $\frac{3}{4} \cdot \frac{7 \cdot 4}{3 \cdot 5}$ and $\frac{7}{5}$ name the same number?

As $\frac{3}{4} \cdot \frac{7 \cdot 4}{3 \cdot 5}$ may be written as $\frac{3 \cdot (7 \cdot 4)}{4 \cdot (3 \cdot 5)}$ we may conclude that the two are equal and that the equation has been solved.

Since $\frac{7 \cdot 4}{3 \cdot 5}$ is the replacement for x that will make $\frac{3}{4} \cdot x = \frac{7}{5}$ a true statement and since we want $\frac{7}{5} \div \frac{3}{4}$ as a solution, it follows that

$$\frac{7}{5} \div \frac{3}{4} = \frac{7 \cdot 4}{3 \cdot 5} .$$

Using the commutative property for multiplication of whole numbers and the definition of multiplication for rational numbers, we get

$$\frac{7}{5} \div \frac{3}{4} = \frac{7}{5} \cdot \frac{4}{3} .$$

This illustrates the common rule used for the division of rational numbers when expressed as fractions.

This method works just as easily in all cases. Let us examine the general case. To solve the equation

$$\frac{a}{b} \cdot x = \frac{c}{d}$$

we think of x as a rational number, say $\frac{u}{v}$. Replacing x with the symbol

$\frac{u}{v}$ and multiplying yields

$$\frac{a \cdot u}{b \cdot v} = \frac{c}{d} .$$

For these two fractions, $\frac{a \cdot u}{b \cdot v}$ and $\frac{c}{d}$, to name the same number, we must have, if possible,

$$(a \cdot u) \cdot d = (b \cdot v) \cdot c.$$

Using the associative and commutative properties gives

$$(a \cdot d) \cdot u = (b \cdot c) \cdot v.$$

Following the method of the special case, we may choose u to be the whole number $(b \cdot c)$ and v to be $(a \cdot d)$. Thus, $\frac{u}{v} = \frac{b \cdot c}{a \cdot d}$.

Is it true that

$$\frac{a}{b} \cdot \frac{(b \cdot c)}{(a \cdot d)} = \frac{c}{d} ?$$

Before you answer this leading question, recall that rational numbers in fractional form have non-zero denominators. This means that $a \cdot d$ must not be zero. As it was implied that $\frac{c}{d}$ is a rational number, $d \neq 0$. We must, therefore, also require that $a \neq 0$. With this restriction, $a \neq 0$, we get the following development:

$$\begin{aligned} \frac{a}{b} \cdot \frac{(b \cdot c)}{(a \cdot d)} &= \frac{a \cdot (b \cdot c)}{b \cdot (a \cdot d)} \\ &= \frac{(a \cdot b) \cdot c}{(b \cdot a) \cdot d} \\ &= \frac{a \cdot b}{b \cdot a} \cdot \frac{c}{d} \\ &= \frac{a \cdot b}{a \cdot b} \cdot \frac{c}{d} \\ &= \frac{a}{a} \cdot \frac{b}{b} \cdot \frac{c}{d} \\ &= 1 \cdot 1 \cdot \frac{c}{d} \\ &= \frac{c}{d} . \end{aligned}$$

Thus, $\frac{a}{b} \cdot \frac{(b \cdot c)}{(a \cdot d)} = \frac{c}{d}$.

Just as $5 \cdot 2 = 10$ conveys the same information as $10 \div 5 = 2$, so does $\frac{a}{b} \cdot \frac{(b \cdot c)}{(a \cdot d)} = \frac{c}{d}$ convey the same information as

$$\frac{c}{d} \div \frac{a}{b} = \frac{b \cdot c}{a \cdot d} .$$

The result $\frac{b \cdot c}{a \cdot d}$ is a rational number. The last equation can, of course, be expressed as

$$\frac{c}{d} \div \frac{a}{b} = \frac{c}{d} \cdot \frac{b}{a} .$$

What has been shown to be true is the well-known rule: To divide one rational number by another when both are expressed in fractional form, invert the divisor and multiply.

Again our development is based upon the use of whole numbers in forming non-negative rational numbers. The procedure for division can, however, be extended to include negative numbers.

Class Exercises

18. Solve each of the following using an analysis similar to that in the text.

(a) $\frac{2}{3} \cdot x = \frac{7}{10}$

(b) $\frac{19}{20} \cdot x = \frac{43}{31}$

(c) $\frac{2}{3} \cdot x = \frac{4}{3}$

(d) $\frac{17}{23} \cdot x = \frac{17}{23}$

(e) $\frac{2}{3} \cdot y = \frac{3}{2}$

19. Which of the following equations have unique solutions?

(a) $\frac{0}{3} \cdot x = \frac{2}{3}$

(b) $\frac{4}{3} \cdot x = \frac{0}{7}$

(c) $\frac{9}{5} \cdot x = \frac{9}{5}$

(d) $\frac{0}{17} \cdot x = \frac{0}{7}$

(e) $4 \cdot x = \frac{7}{9}$

(f) $\frac{5}{8} \cdot x = 4$

6.8 Operations on the Integers

Thus far we have restricted our discussion of the operations on rationals to non-negative rational numbers. The reason for this is that most seventh grade students will study computations using the numbers of arithmetic (the non-negative rationals) well before they meet the set of integers or the negative rationals. The teacher, however, should see that the definitions and rules thus far established necessarily must apply to all rational numbers, positive, zero, and negative. The extension is easy once the operating rules for the integers are established.

Recall that in Chapter 5 the set of integers was shown to contain the set of counting numbers, zero, and the set of opposites of the counting numbers. These subsets of the set of integers were called the positive integers, zero, and the negative integers, respectively. In symbols we can represent the set of integers as:

$$I = \{\ldots, {}^-3, {}^-2, {}^-1, 0, 1, 2, 3, \ldots\}.$$

The identity element for addition using the set of integers is 0 since for every integer $\underline{a}$, $a + 0 = 0 + a = a$. In the set of integers each element also has an opposite called its additive inverse. In Chapter 4 two elements were defined as inverses of each other under a given binary operation if the result of this operation on the two elements is the identity element for that operation. Thus, additive inverses for the integers are integers which when added give the identity element, 0. Note in the examples how the opposites serve as additive inverses.

$$2 + {}^-2 = 0$$
$${}^-5 + 5 = 0$$
$$0 + 0 = 0$$

The operation of addition with integers was introduced in the last chapter using the number line. Subtraction can be handled in much the same way by making use of the fact that if a and b are integers, then $a - b = a + {}^-b$. This property of subtraction allows us to change every subtraction problem into an addition problem. For example:

$$\begin{aligned} 7 - 11 &= 7 + ({}^-11) = {}^-4 \\ ({}^-7) - 11 &= ({}^-7) + ({}^-11) = {}^-18 \\ 7 - ({}^-11) &= 7 + 11 = 18 \\ ({}^-7) - ({}^-11) &= ({}^-7) + 11 = 4 \end{aligned}$$

In general, subtracting a number is equivalent to adding its additive inverse.

Consider next the multiplication of integers. We know from the properties of whole numbers that the product of two positive integers is a positive integer. We also know that the product of zero and any integer is zero. But how should we define the product of a positive and a negative integer?

The product $4 \cdot ({}^-7)$ may be expressed as the sum $({}^-7) + ({}^-7) + ({}^-7) + ({}^-7)$ which we know equals ${}^-28$. Also, since we want the commutative property to hold, we will agree that $4 \cdot ({}^-7)$ and $({}^-7) \cdot 4$ mean the same integer. That is,

$$4 \cdot ({}^-7) = ({}^-7) \cdot 4.$$

Another way to evaluate $4 \cdot ({}^-7)$ is illustrated here. From the property of additive inverses we know that

$$7 + ({}^-7) = 0.$$

Hence, we may write

$$4 \cdot [7 + (^{-}7)] = 4 \cdot 0 = 0.$$

Using the distributive property for integers, we then get

$$4 \cdot (7) + 4 \cdot (^{-}7) = 0.$$

Since $4 \cdot (7)$ and $4 \cdot (^{-}7)$ add to zero, they must be additive inverses or opposites. But $4 \cdot (7)$ is 28. Now since $4 \cdot (^{-}7)$ is the additive inverse of $4 \cdot (7)$, it must be the additive inverse of 28. Thus, we conclude that $4 \cdot (^{-}7) = {}^{-}28$.

The two methods shown give the same results. In general, we can say that the product of a positive integer and a negative integer is a negative integer.

What meaning should we give to $(^{-}4) \cdot (^{-}7)$? Proceeding as before we get the following:

$$7 + (^{-}7) = 0$$
$$(^{-}4) \cdot [7 + (^{-}7)] = (^{-}4) \cdot 0 = 0$$
$$(-4) \cdot 7 + (^{-}4) \cdot (^{-}7) = 0.$$

Since $(^{-}4) \cdot 7$ and $(^{-}4) \cdot (^{-}7)$ add to zero, they are additive inverses. But $(^{-}4) \cdot 7 = {}^{-}28$. Thus, we conclude that $(^{-}4) \cdot (^{-}7)$ is the additive inverse of ${}^{-}28$, or $(^{-}4) \cdot (^{-}7) = 28$. The same development will hold for any two negative integers. In general, we say that the product of two negative integers is a positive integer.

An interesting and informal introduction to the product of integers using patterns in a multiplication table is given in the SMSG publication, Mathematics for Junior High School, Vol. 2. The procedure for division of integers follows directly from the multiplication procedure. If two positive or two negative integers are divided, the quotient is positive. If a positive and a negative integer are divided, the quotient is negative.

Class Exercises

20. Do each problem using the fact that $a - b = a + (^{-}b)$.

(a) $17 - 21$
(b) $(^{-}17) - 21$
(c) $17 - (^{-}21)$
(d) $(^{-}17) - (^{-}21)$

21. Evaluate each product.

(a) $7 \cdot 13$
(b) $(^{-}7) \cdot 13$
(c) $7 \cdot (^{-}13)$
(d) $(^{-}7) \cdot (^{-}13)$

22. Evaluate each quotient.

(a) $81 \div 3$ (c) $81 \div ({}^{-}3)$

(b) $({}^{-}81) \div 3$ (d) $({}^{-}81) \div ({}^{-}3)$

Let us look again at the operations this time using negative rational numbers. In general, the definition of the four basic operations and the properties developed in this chapter for the non-negative rational numbers can be extended to include also the negative rational numbers. In so doing, however, we will make use of a modified definition of rational numbers. We may define the set of rational numbers as all numbers that can be expressed in the form $\frac{a}{b}$ where a is an integer and b is a counting number. This change in definition now admits the negative rational numbers. The corresponding change in the equation definition would be that the rational numbers is the set of all solutions of equations in the form $bx = a$ where a is an integer and b a counting number.

With this change we can now ask for solutions of equations such as

$$4x = {}^{-}3$$

and know that they will be rational numbers. The solution here is ${}^{-}(\frac{3}{4})$, the additive inverse of $\frac{3}{4}$. The solution may be written as $\frac{{}^{-}3}{4}$ with the numerator of the fraction a negative integer.

The properties previously established for the non-negative rational numbers will hold for the negative rationals as well. Likewise, the definitions of the four fundamental operations apply to all rational numbers through the properties of integers. In the following examples study how the operations involving negative rational numbers have been completed by making use of our knowledge of integers.

$$\frac{1}{2} + \frac{{}^{-}3}{4} = \frac{1 \cdot 4 + 2 \cdot ({}^{-}3)}{2 \cdot 4} = \frac{4 + ({}^{-}6)}{8} = \frac{{}^{-}1}{4}$$

$$\frac{{}^{-}2}{3} - \frac{3}{4} = \frac{{}^{-}2}{3} + \frac{{}^{-}3}{4} = \frac{({}^{-}2) \cdot 4 + 3 \cdot ({}^{-}3)}{4 \cdot 3} = \frac{({}^{-}8) + ({}^{-}9)}{12} = \frac{{}^{-}17}{12}$$

$$\frac{{}^{-}1}{5} \cdot \frac{{}^{-}3}{4} = \frac{({}^{-}1) \cdot ({}^{-}3)}{5 \cdot 4} = \frac{3}{20}$$

$$\frac{{}^{-}2}{5} \div \frac{3}{4} = \frac{{}^{-}2}{5} \cdot \frac{4}{3} = \frac{({}^{-}2) \cdot 4}{5 \cdot 3} = \frac{{}^{-}8}{15}$$

Chapter Exercises

1. Evaluate:

   (a) $\frac{7}{8} + \frac{11}{15}$

   (b) $\frac{7}{8} - \frac{11}{15}$

   (c) $\frac{7}{8} \cdot \frac{11}{15}$

   (d) $\frac{7}{8} \div \frac{11}{15}$

2. Find the following sums:

   (a) $1 + \frac{1}{2}$

   (b) $1 + \frac{1}{2} + \frac{1}{4}$

   (c) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$

   (d) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$

3. Use the results of Exercise 2 to make an informal guess of the following sums:

   (a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{32} + \frac{1}{64}$

   (b) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{512} + \frac{1}{1024}$

4. Find $(\frac{3}{4} \div \frac{9}{10}) \div \frac{1}{2}$ and $\frac{3}{4} \div (\frac{9}{10} \div \frac{1}{2})$. Are the answers the same? What conclusion can be drawn from the last answer?

5. Find $\frac{3}{4} \div (\frac{2}{3} - \frac{1}{2})$ and $(\frac{3}{4} \div \frac{2}{3}) - (\frac{3}{4} \div \frac{1}{2})$. Are the answers the same?

6. Using the rational numbers $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$, prove the associative property for multiplication.

7. Evaluate:

   (a) $9 - 13$

   (b) $9 - (^-13)$

   (c) $(^-9) - 13$

   (d) $(^-9) - (^-13)$

8. Evaluate:

   (a) $(\frac{^-9}{4}) - \frac{3}{2}$

   (b) $\frac{8}{7} + (\frac{^-3}{4})$

   (c) $(\frac{^-1}{2}) \cdot (\frac{^-3}{4})$

   (d) $(\frac{^-2}{3}) \div \frac{5}{6}$

9. The sums $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$ and $\frac{3}{5} = \frac{1}{2} + \frac{1}{10}$ are examples of fractions written as the sum of unit fractions, i.e., fractions with numerator 1. Represent each of the following rational numbers as sums of unit fractions. A particular unit fraction may be used only once in each sum.

   (a) $\frac{13}{12}$

   (b) $\frac{8}{15}$

   (c) $\frac{9}{20}$

   (d) $\frac{47}{60}$

10. (a) Show that $\frac{1}{1 \cdot 2} = \frac{1}{1} - \frac{1}{2}$.

(b) Show that $\frac{1}{2 \cdot 3} = \frac{1}{2} - \frac{1}{3}$.

(c) Show that $\frac{1}{4 \cdot 5} = \frac{1}{4} - \frac{1}{5}$.

(d) Express $\frac{1}{9 \cdot 10}$ as the difference of two unit fractions.

(e) Express $\frac{1}{18 \cdot 19}$ as the difference of two unit fractions.

(f) Use the results of the above to find the sum
$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{18 \cdot 19}$.

11. Find the sums in (a), (b), and (c):

(a) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3}$

(b) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4}$

(c) $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5}$

(d) Make an educated guess as to the sum:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{18 \cdot 19} .$$

(e) Make an educated guess as to the sum:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{99 \cdot 100} .$$

12. Solve the equation

$$\frac{3}{5} \cdot x + \frac{3}{16} = \frac{2}{3} + \frac{1}{4} .$$

13. A "magic square" is a square array of numbers such that each row, each column, and the two diagonals all add to the same number. Complete the table below to form a 3×3 "magic square."

$\frac{2}{3}$		
$\frac{1}{4}$		
$\frac{1}{3}$	$\frac{3}{4}$	$\frac{1}{6}$

Answers to Class Exercises

1. (a) $\frac{19}{15}$ (b) $\frac{42}{16}$ (c) $\frac{4}{4}$ (d) $\frac{6}{9}$ (e) $\frac{9}{18}$. (f) $\frac{8}{8}$

2. No, in the sense that we have different fractional representations of the same rational numbers.

$$\frac{4}{4} = 1 \quad \text{and} \quad \frac{6}{9} = \frac{2}{3}$$

3. (a) $\frac{7}{1}$ (b) Yes, the rational number $\frac{7}{1}$ is a name for the solution of the equation $1x = 7$ which also has a solution named 7. We have agreed therefore that $\frac{7}{1}$ and 7 name the same number.

4. (a) $\frac{126}{14}$. (b) Yes, the number $\frac{126}{14}$ is the solution of $14y = 126$ which also has a solution $y = 9$. Thus, following our agreement, 9 and $\frac{126}{14}$ are names for the same rational number.

5. (a) $\frac{155}{116}$ (b) $\frac{155}{116}$ (c) $\frac{606}{505}$ (d) $\frac{606}{505}$

(e) They are the same, illustrating the commutative property.

(f) They are the same. Since $\frac{606}{505} = \frac{6 \cdot 101}{5 \cdot 101} = \frac{6}{5}$ this illustrates the identity property of 0.

6. (a) $\frac{17}{12}$ (b) $\frac{242}{120}$ (c) $\frac{38}{30}$ (d) $\frac{242}{120}$

(e) They are the same, illustrating the associative property.

7. $\frac{165}{15}$. $6 + 5 = 11$. The symbols 11 and $\frac{165}{15}$ both designate the answer to $15z = 165$.

8. If $\frac{1}{2} + \frac{1}{2}$ is computed this way, one obtains $\frac{2}{4}$ which would not satisfy one's intuition. Also, the distributive law would clearly fail to hold.

9. $\frac{12}{1}$. Both $\frac{12}{1}$ and $12 = 4 \cdot 3$ name the solution of $1x = 12$.

10. $\frac{280}{14}$. Yes, $\frac{28}{7} \cdot \frac{10}{2} = \frac{280}{14} = \frac{14 \cdot 20}{14 \cdot 1} = \frac{20}{1} = 20$. Also, $\frac{28}{7} = 4$, $\frac{10}{2} = 5$, and $4 \cdot 5 = 20$.

11. The result of both computations is $\frac{216}{308}$.

12. The left-hand side is $\frac{465}{308}$ and the right-hand side is $\frac{1860}{1232}$. The two may be shown to be equal by computing $465 \cdot 1232$ and $1860 \cdot 308$ or by observing $\frac{1860}{1232} = \frac{4 \cdot 465}{4 \cdot 308} = \frac{465}{308}$.

13. $\frac{56}{15}$

14. (a) $\frac{12}{12}$ (b) $\frac{90}{90}$ (c) $\frac{10}{10}$ (d) $\frac{63}{63}$

What interesting fact is observed?

15. Yes, for both 1 and $\frac{96}{96}$ are names of the solution of $96w = 96$.

16. (a) $\frac{49}{68}$ (c) $\frac{0}{a}$, a any counting number

(b) $\frac{2}{15}$ (d) $\frac{0}{a}$, a any counting number.

For (c) and (d), 0 would also be correct.

17. Yes, each method relies on the subtraction of whole numbers.

18. (a) $\frac{21}{20}$ (b) $\frac{860}{589}$ (c) $\frac{12}{6}$ or $\frac{2}{1}$ (d) $\frac{1}{1}$ (e) $\frac{9}{4}$

19. (b), (c), (e), and (f).

20. (a) $^{-}4$ (b) $^{-}38$ (c) 38 (d) 4

21. (a) 91 (b) $^{-}91$ (c) $^{-}91$ (d) 91

22. (a) 27 (b) $^{-}27$ (c) $^{-}27$ (d) 27

ANSWERS TO CHAPTER EXERCISES

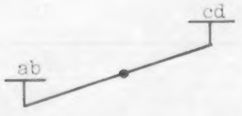
Answers to Problems in the Introduction

1.
$$\begin{array}{rcr} \text{SEND} & \text{——} & 9567 \\ +\ \text{MORE} & \text{——} & 1085 \\ \hline \text{MONEY} & \text{——} & 10652 \end{array}$$

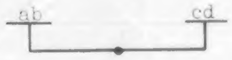
2. The winning strategy for the sample number game is to always choose a number so that the total after your turn will always be a multiple of seven.

   The second number game, where the winning sum is 85, can be won by the player reaching 77 first. Regardless of what number his opponent then calls, he will always be able to choose a number that will make the sum 85. The magic number is eight, one more than the largest number that can be used. Since $85 = 8 \cdot 10 + 5$ all critical points are 5 more than multiples of 8 ; i.e., $77 = 72 + 5$, $69 = 64 + 5$, and so forth. The first player may win by choosing 5 and then each time afterward, picking a number to make the total 5 more than a multiple of eight; i.e, 13, 21, 29, ...

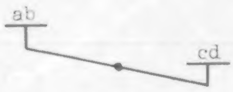
3. The problem of the counterfeit coin among six coins differs from the heavy marble problem in that weighing three coins against three coins will yield no new information. From the fact that the scale does not balance we only learn that the coins are not all of the same weight. To do this problem it will be convenient to label the coins a, b, c, d, e, f. Suppose we weight a, b against c, d. There are three possible outcomes of such a weighing.



Case I



Case II

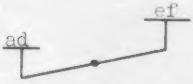


Case III

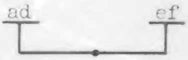
Case I

Here we see that either a or b is heavy or c or d is light, while e and f are good.

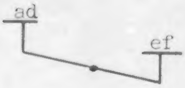
Second weighing -- compare a, d with e, f. Again three outcomes are possible.



thus a is heavy
(problem solved)

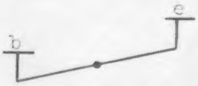


thus b is heavy
or c is light
(third weighing needed)

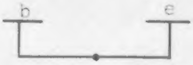


thus d is light
(problem solved)

Third weighing -- compare b and e -- two possibilities exist.



thus b is heavy
(problem solved)

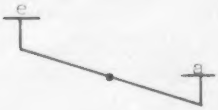


thus c is light
(problem solved)

Case II

When a, b and c, d balance, either e or f is the false coin.

Second weighing -- compare e and a . Three possibilities exist.

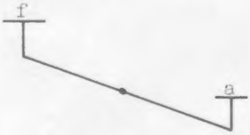


thus e is light
(problem solved)

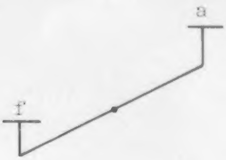
thus f is the counterfeit coin (third weighing needed to determine if heavy or light.)

thus e is heavy
(problem solved)

Third weighing -- compare f and a. Two possibilities exist.



thus f is light
(problem solved)



thus f is heavy
(problem solved)

Case III

This case is handled with the same reasoning as Case I with the obvious changes. Start with a or b in light and c or d is heavy. Notice that the key to the solutions of both Case I and Case III rests on the interchanging of the position of a coin from one side to the other.

4. Only figures b and c may be traced. Figures a and d each have more than two odd vertices.

5. It is not possible to connect the utilities and houses under the conditions stated. This is discussed further in Chapter 10.

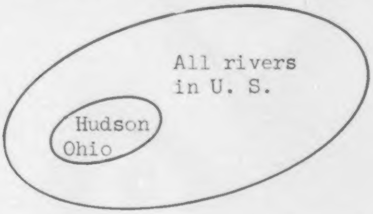
6. The smallest possible number of weights needed to weigh objects, in pounds, between 1 pound and 63 pounds is six; namely, weights of 1, 2, 4, 8, 16, and 32 pounds. A similar problem occurs again in Chapter 2.

Answers to Chapter Exercises

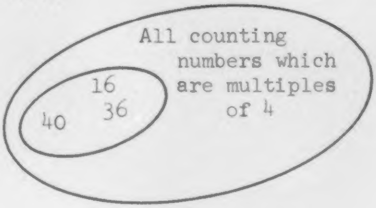
Chapter 1: Math for Junior High Teachers

1. {4}, {5}, {6}, {4,5}, {4,6}, {5,6}, {4,5,6}, ∅.

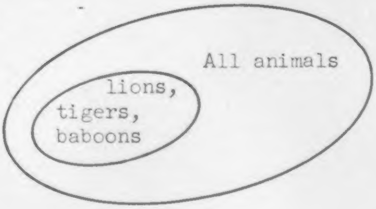
2. (a)



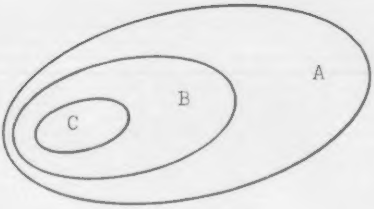
(c)



(b)



3. Yes.



4. (a) $A \cap B = \{\text{girl}, \text{chair}\}$

(b) $A \cap C = \{\text{chair}\}$ and $C \cap A = \{\text{chair}\}$

(c)
$$\begin{aligned} A \cap (B \cup C) &= \{\text{boy}, \text{girl}, \text{chair}\} \cap (\{\text{girl}, \text{chair}, \text{dog}\} \cup \{\text{chair}, \text{dog}, \text{cat}\}) \\ &= \{\text{boy}, \text{girl}, \text{chair}\} \cap \{\text{girl}, \text{chair}, \text{dog}, \text{cat}\} \\ &= \{\text{girl}, \text{chair}\} \end{aligned}$$

also
$$\begin{aligned} (A \cap B) \cup (A \cap C) &= (\{\text{boy}, \text{girl}, \text{chair}\} \cap \{\text{girl}, \text{chair}, \text{dog}\}) \cup (\{\text{boy}, \\ &\qquad \text{girl}, \text{chair}\} \cap \{\text{chair}, \text{dog}, \text{cat}\}) \\ &= \{\text{girl}, \text{chair}\} \cup \{\text{chair}\} \\ &= \{\text{girl}, \text{chair}\} \end{aligned}$$

(d)
$$\begin{aligned} A \cap (B \cap C) &= \{\text{boy}, \text{girl}, \text{chair}\} \cap (\{\text{girl}, \text{chair}, \text{dog}\} \cap \{\text{chair}, \text{dog}, \text{cat}\}) \\ &= \{\text{boy}, \text{girl}, \text{chair}\} \cap \{\text{chair}, \text{dog}\} \\ &= \{\text{chair}\} \end{aligned}$$

also
$$\begin{aligned} C \cap (A \cap B) &= \{\text{chair}, \text{dog}, \text{cat}\} \cap (\{\text{boy}, \text{girl}, \text{chair}\} \cap \{\text{girl}, \text{chair}, \text{dog}\}) \\ &= \{\text{chair}, \text{dog}, \text{cat}\} \cap \{\text{girl}, \text{chair}\} \\ &= \{\text{chair}\} \end{aligned}$$

5. (a) Yes.

$$\emptyset \cup H = H$$
$$H \cup \emptyset = H$$

Therefore, $\emptyset \cup H = H \cup \emptyset$.

(b) No. $\emptyset \cap H = \emptyset$. $\emptyset$ has no elements in common with H.

6. (a) Yes. The counting numbers are composed of the even numbers and the odd numbers.

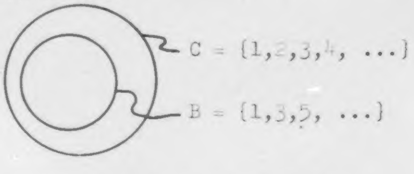
(b) Yes. Same reason as (a).

(c) No. A and B are disjoint subsets of C.

(d) Yes. The set of elements in A or B is the same as those in B or A.

(e) No. Same reason as (c).

(f)



(g) No. They do not contain the same elements.

7. (a) F = {1,3,5,7,9, ...} . F is the set of odd counting numbers.

   (b) ∅

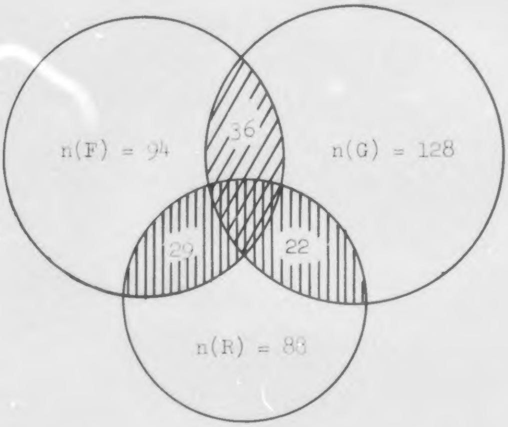
   (c) Yes. Set F could be, for example, the same as set C.

8. (a) Yes. Every element of A is an element of B.

   (b) Yes. Same reason as (a).

9. No. You can look for empty spaces.

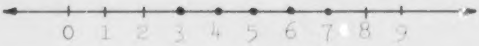
10.



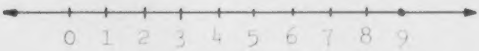
The diagram simplifies our bookkeeping and also shows that the intersection of the sets is included three times but should not be subtracted three times.

11. The columns on a calendar establish such a correspondence.

12. (a) {3,4,5,6,7}

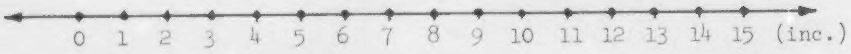


   (b) {9}

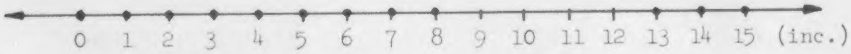


12. (continued)

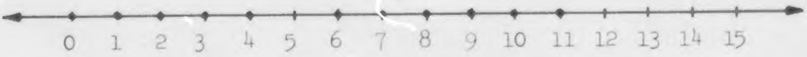
(c) $\{0,1,2,3, \ldots\}$, the set of whole numbers.



(d) The set of all whole numbers except 9, 10, 11, 12.



(e) $\{0,1,2,3,4,6,7,8,9,10,11\}$, the set of all whole numbers less than 12 except 5.



13. $$\begin{array}{llllllllll} W = \{0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & \ldots\} \\ \quad\updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & & & & \text{etc.} \\ O = \{1, & 3, & 5, & 7, & 9, & 11, & 13, & 15, & 17, & \ldots\} \end{array}$$

Set O is a proper subset of set W, yet for any member of W we may find a corresponding member of O, and for any member of O we may find a corresponding member of W. In other words, given an infinite set, a proper subset (which is itself infinite) may be put into a one-to-one correspondence with the given set.

Answers to Chapter Exercises

Chapter 2: Math for Junior High School Teachers

1. (a) ∩|||||||||

   34_{five}

   25_{seven}

   (b) ∩∩∩∩∩ |||

   203_{five}

   104_{seven}

   (c) 𓍢𓍢𓍢𓍢𓍢𓍢 ∩∩∩∩∩∩||||||

   10131_{five}

   1641_{seven}

   (d) 𓆼 𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢𓍢 ∩∩∩∩∩∩

   30320_{five}

   5500_{seven}

2. (a) $(1 \times 5^2) + (0 \times 5) + (0 \times 1)$

   (b) $(1 \times 2^3) + (1 \times 2^2) + (1 \times 2) + (0 \times 1)$

   (c) $(2 \times 3^2) + (0 \times 3) + (1 \times 1)$

   (d) $(1 \times 4^2) + (1 \times 4) + (1 \times 1)$

   (e) $(1 \times 8) + (7 \times 1)$

   (f) $(T \times 1)$ or (10×1)

   (g) $(1 \times 10^2) + (0 \times 10) + (0 \times 1)$

   (h) $(1 \times 7^2) + (0 \times 7) + (0 \times 1)$

3. (a) 44_{five} (e) 16_{eight}

(b) 1101_{two} (f) 9_{twelve}

(c) 200_{three} (g) 99

(d) 103_{four} (h) 66_{seven}

4. (a) 1 2 2 122_{five} (d) 0 3 0 30_{five}

(b) 0 4 4 44_{five} (e) 2 2 3 223_{five}

(c) 3 4 4 344_{five} (f) 4 4 4 444_{five}

5.

Base ten Numerals	Base five Numerals	Base ten Numerals	Base five Numerals
1	/	14	∠□
2	∠	15	△○
3	△	16	△/
4	□	17	△∠
5	/○	18	△△
6	//	19	△□
7	/∠	20	□○
8	/△	21	□/
9	/□	22	□∠
10	∠○	23	□△
11	∠/	24	□□
12	∠∠	25	/○○
13	∠△		

6. (a) 294 (b) 193 (c) 1511 (d) 525

7. (a) 3145_{six} (b) 101100_{two} (c) $10E_{twelve}$ (d) 11031_{four}

8. 73_{eight}

Answers to Chapter Exercises

Chapter 3: Math for Junior High School Teachers

1. (a) 432_{eight} (d) 710_{eight}
   (b) 1337_{eight} (e) 303_{eight}
   (c) 404_{eight}

2. (a) 40_{eight} (b) 611_{eight} (c) 12_{eight}

3. (a) 3237_{eight} (b) 34136_{eight}

4. 7_{eight}

   362_{eight}

5. (a) $9 = 72 \div 8$ or $8 = 72 \div 9$
   $2 = 80 \div 40$ $40 = 80 \div 2$
   $5 = 25 \div n$ $n = 25 \div 5$
   $4 = 24 \div n$ $n = 24 \div n$
   $n = 100 \div 10$ $10 = 100 \div n$

   (b) $3_{five} = 11_{five} \div 2_{five}$ or $2_{five} = 11_{five} \div 3_{five}$
   $n_{five} = 31_{five} \div 4_{five}$ or $4_{five} = 31_{five} + n_{five}$

   (c) $4_{seven} = 26_{seven} \div 5_{seven}$ or $5_{seven} = 26_{seven} \div 4_{seven}$
   $4_{seven} = 33_{seven} \div n_{seven}$ or $n_{seven} = 33_{seven} \div 4_{seven}$
   $n_{seven} = 100_{seven} \div n_{seven}$

6. 6_{seven}, 10_{seven}

7. (a) base five (d) base five
   (b) base seven (e) any base > 3
   (c) base seven (f) any base > 8

8. (a) 0

   (b) 0, 1, or 2. Evenness cannot be recognized by the last digit in base three.

9. (a) 0 or 1 (b) 0 or 1

10. It is not possible under the conditions stated, since N + T = N implies T = zero and E + H = E implies H = zero. Both cannot be zero, under the conditions stated.

11. (a) 140_{seven} (b) 3_{seven} (c) 462_{seven}

Answers to Chapter Exercises

Chapter 4: Math for Junior High School Teachers

1.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

2. (a) 1 (b) 2 (c) 2 (d) 1 (e) 3

3. (a) yes

(b) yes

(c) yes

(d) The inverse of 1 is 1, of 3 is 3.

(e) Not necessarily. 2 × 2 = 0.

4. (a) Both are commutative.

(b) Yes, 0

(c) yes

(d) No. 0 + (E × 0) = 0 + E = 0, but (0 + E) × (0 + 0) = 0 × E = E

5. Not closed; $2 \cdot 25 + 3 = 53$ and 53 is not in the set.
Not commutative.
Not associative.

6. Closed.
Not commutative.
Not associative.

7. (a)

ANTH	I	V	H	R
I	I	V	H	R
V	V	I	R	H
H	H	R	I	V
R	R	H	V	I

(b) Yes. (c) Yes. (d) Yes. (e) Yes, I. (f) Yes. Each element is its own inverse.

8. (a) Yes.

$$A \cup (B \cap C) = \{1,2,3,4,5\} \cup \{3,5,8\} = \{1,2,3,4,5,8\}$$

$$(A \cup B) \cap (A \cup C) = \{1,2,3,4,5,7,8\} \cap \{1,2,3,4,5,8,9\} = \{1,2,3,4,5,8\}$$

(b) Yes.

$$A \cap (B \cup C) = \{1,2,3,4,5\} \cap \{1,3,4,5,7,8,9\} = \{1,3,4,5\}$$

$$(A \cap B) \cup (A \cap C) = \{3,4,5\} \cup \{1,3,5\} = \{1,3,4,5\}$$

Answers to Chapter Exercises

Chapter 5: Math for Junior High School Teachers

1. This may be shown directly by simply computing both sides. A more interesting and illuminating method is through the use of the distributive law.

$$\begin{aligned} 4(5 + 6 + 7 + 8) &= 4((5 + 6 + 7) + 8) \\ &= 4(5 + 6 + 7) + (4 \cdot 8) \\ &= 4((5 + 6) + 7) + (4 \cdot 8) \\ &= 4(5 + 6) + (4 \cdot 7) + (4 \cdot 8) \\ &= (4 \cdot 5) + (4 \cdot 6) + (4 \cdot 7) + (4 \cdot 8) \end{aligned}$$

2. This may also be shown directly by computation but it is instructive to see a method using the associative law.

$$\begin{aligned} \Big(\big((3 + 4) + 9\big) + 5\Big) + 2 &= \big((3 + 4) + 9\big) + (5 + 2) \\ &= (3 + 4) + \big(9 + (5 + 2)\big) \\ &= 3 + \Big(4 + \big(9 + (5 + 2)\big)\Big) \end{aligned}$$

3. There are many answers: $\frac{102}{128}$ and $\frac{459}{576}$ are two possible answers.

4. There is nothing imporper about improper fractions. An improper fraction, numerator greater than the denominator, may be written in another form, e.g., $\frac{19}{3}$ as $6\frac{1}{3}$. This second form may serve a useful psychological purpose and may be better suited to some calculations or commerce, but serves no useful mathematical purpose. Indeed there are situations in which such proper forms are a hindrance.

5. Yes. The symbol $\frac{11}{5}$ names a solution to $5x = 11$. Since it can be shown that the equations $5x = 11$ and $35x = 77$ are the same, $\frac{11}{5}$ must also name the solution of $35x = 77$.

   No. The symbol $\frac{7}{9}$ names a solution to $9x = 7$. Since $9x = 7$ and $21x = 27$ are not the same equation, $\frac{7}{9}$ does not name a solution of $21x = 27$.

6. In each care apply the definition for equivalent fractions.

7. 6, 9, 103.

8. Yes, in each case. The counting numbers are 3, 2, 31, and a. Observe, for example, that while $\frac{6}{2}$ is an answer of $2x = 6$, so also in 3 a solution of this equation.

9. $\frac{3}{17}$, $\frac{3}{8}$, $\frac{3}{6}$, $\frac{3}{5}$, $\frac{3}{2}$, $\frac{3}{1}$.

10. $\frac{2}{7}$, $\frac{5}{7}$, $\frac{6}{7}$, $\frac{7}{7}$, $\frac{9}{7}$, $\frac{14}{7}$, $\frac{18}{7}$.

11. $\frac{8}{9}$, $\frac{11}{12}$, $\frac{14}{15}$, $\frac{19}{20}$, $\frac{99}{100}$

12. b, d, e.

Answers to Chapter Exercises

Chapter 6: Math for Junior High School Teachers

1. (a) $\frac{193}{120}$ (b) $\frac{17}{120}$ (c) $\frac{77}{120}$ (d) $\frac{105}{88}$

2. (a) $\frac{3}{2}$ (b) $\frac{7}{4}$ (c) $\frac{15}{8}$ (d) $\frac{31}{16}$

3. (a) $\frac{127}{64}$ (b) $\frac{2047}{1024}$

4. $\frac{60}{36}$, $\frac{30}{72}$. The answers are not the same which shows that division is not associative.

5. $\frac{18}{4}$, $\frac{^-3}{8}$. The answers are not the same which shows that division is not distributive over subtraction.

6. The problem is to show that the two expressions, $(\frac{a}{b} \cdot \frac{c}{d}) \cdot \frac{e}{f}$ and $\frac{a}{b} \cdot (\frac{c}{d} \cdot \frac{e}{f})$ are equal.

$$\begin{aligned}(\frac{a}{b} \cdot \frac{c}{d}) \cdot \frac{e}{f} &= (\frac{ac}{bd}) \cdot \frac{e}{f} \\ &= \frac{ace}{bdf} .\end{aligned}$$

Also,

$$\begin{aligned}\frac{a}{b} \cdot (\frac{c}{d} \cdot \frac{e}{f}) &= \frac{a}{b} \cdot (\frac{ce}{df}) \\ &= \frac{ace}{bdf} .\end{aligned}$$

Thus, since the two expressions are each equal to $\frac{ace}{bdf}$ we see that the associative law does hold for multiplication of rational numbers.

7. (a) $^-4$ (b) 22 (c) $^-22$ (d) 4

8. (a) $\frac{^-30}{8}$ (b) $\frac{20}{28}$ (c) $\frac{3}{8}$ (d) $^-\frac{12}{15}$

9. (a) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ (c) $\frac{1}{4} + \frac{1}{5}$

(b) $\frac{1}{3} + \frac{1}{5}$ (d) $\frac{1}{2} + \frac{1}{4} + \frac{1}{30}$

10. (a) $\frac{1}{1 \cdot 2} = \frac{1}{2}$ and $\frac{1}{1} - \frac{1}{2} = \frac{2 - 1}{2} = \frac{1}{2}$

(b) $\frac{1}{2 \cdot 3} = \frac{1}{6}$ and $\frac{1}{2} - \frac{1}{3} = \frac{3 - 2}{6} = \frac{1}{6}$

(c) $\frac{1}{4 \cdot 5} = \frac{1}{20}$ and $\frac{1}{4} - \frac{1}{5} = \frac{5 - 4}{20} = \frac{1}{20}$

(d) $\frac{1}{9 \cdot 10} = \frac{1}{9} - \frac{1}{10}$

(e) $\frac{1}{18 \cdot 19} = \frac{1}{18} - \frac{1}{19}$

(f)
$$\begin{aligned} \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{18 \cdot 19} &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{18} - \frac{1}{19}\right) \\ &= \frac{1}{1}\left({}^{-}\frac{1}{2} + \frac{1}{2}\right) + \left({}^{-}\frac{1}{3} + \frac{1}{3}\right) + \ldots + \left({}^{-}\frac{1}{18} + \frac{1}{18}\right) - \frac{1}{19} \\ &= \frac{1}{1} - \frac{1}{19} \\ &= \frac{18}{19} \end{aligned}$$

11. (a) $\frac{2}{3}$ (b) $\frac{3}{4}$ (c) $\frac{4}{5}$ (d) $\frac{18}{19}$ (e) $\frac{99}{100}$

12. $\frac{175}{144}$

13.

$\frac{2}{3}$	$\frac{1}{12}$	$\frac{1}{2}$
$\frac{1}{4}$	$\frac{5}{12}$	$\frac{7}{12}$
$\frac{1}{3}$	$\frac{3}{4}$	$\frac{1}{6}$